THE

MATHEMATICAL GAZETTE

EDITED BY

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MARGARET PUNNETT, 1867-1946.

Honorary Secretary, 1912–1939.

MISS MARGARET PUNNETT's work as a trainer of teachers began with a short time at Saffron Walden Training College, followed by three years as Principal of the Cambridge Training College for Women, and in 1902 she became Vice-Principal of the London Day Training College (from 1932 the University of London Institute of Education), a post she held until retirement in 1933. As a faithful colleague and disciple of Sir Percy Nunn, she played a steady part in his great work at the London Day Training College. Her book, *The Groundwork of Arithmetic* (1914), was a sane and inspiring course for young children, of great benefit to the novice-teacher.

Miss Punnett joined our Association in 1910 and in 1912 was elected the first woman secretary of the Association, serving as joint secretary with the late Charles Pendlebury till his retirement (after fifty years' service) in 1936; on her own retirement from this office in 1939 she was elected a Vice-President. The services of a Secretary, so vital and yet so difficult to describe in detail, are seldom sufficiently appreciated by the ordinary member; Council members have more opportunity of estimating the debt the Association owes to its succession of Secretaries, and Miss Punnett's share in that debt is not small. We recall her imperturbable competence and her quiet kindness, the competence which was never ostentatious, the kindly manner which was never a cloak for weakness or indecision. Co-operating firmly on general policy with Pendlebury, she never failed to keep the Council reminded of its duties to women as well as to men teachers. We record here our deep gratitude for services so long continued and so faithfully rendered.

GENERALISED METRICAL THEOREMS.

By S. Vajda.

It is well known that Euclidean metric geometry is a special case of projective geometry, inasmuch as it refers to transformations which leave the line at infinity and the circular points invariant. Every projective theorem will thus have one or more Euclidean counterparts according to which line and which points on it are taken to be the line at infinity and the circular points. It is, on the other hand, also possible to translate any theorem into projective language by removing every trace of metrical properties. A theorem of projective geometry is thus obtained of which the original theorem is a special case. This principle will be illustrated by applying it to the following elementary

Theorem A. The symmetrics of the orthocentre of a triangle with respect to its sides lie on the circumcircle (Fig. 1).

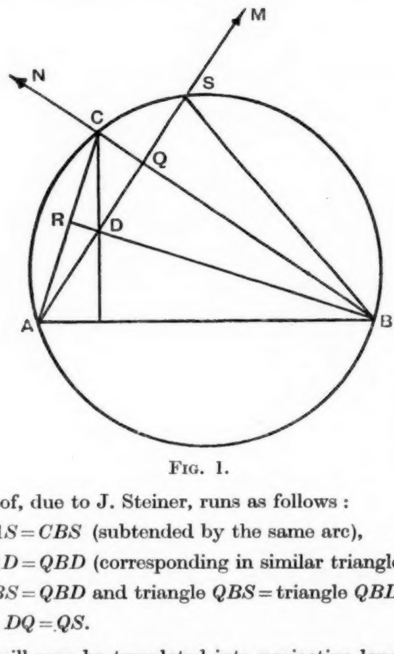

Fig. 1.

An elegant proof, due to J. Steiner, runs as follows:

$CAS = CBS$ (subtended by the same arc),

$RAD = QBD$ (corresponding in similar triangles).

Hence $QBS = QBD$ and triangle QBS = triangle QBD.

It follows that $DQ = QS$.

This theorem will now be translated into projective language. The circle through A, C, S, B will be spoken of as a conic through the circular points I and J on the line at infinity of the metric plane. But the circular points themselves must be described as the double points of an involution. This latter must, in Euclidean language, be the orthogonal one. Now an involution is defined by two pairs of corresponding elements. The orthogonal involution will thus be described as the involution generated on the line at infinity, l, by the pencil of conics through A, B, C and D. This is indeed the

involution looked for, because the (degenerate) conics (AB, CD) and (AD, CB) consist each of a pair of orthogonal lines.

It remains to express the fact that DQ equals QS in projective terms. This is equivalent to stating that $\{D, S, Q, M\} = -1$, *i.e.* that the points D, S, Q and M form a harmonic range, where M is the point of intersection of l and AD. We thus obtain (Fig. 2):

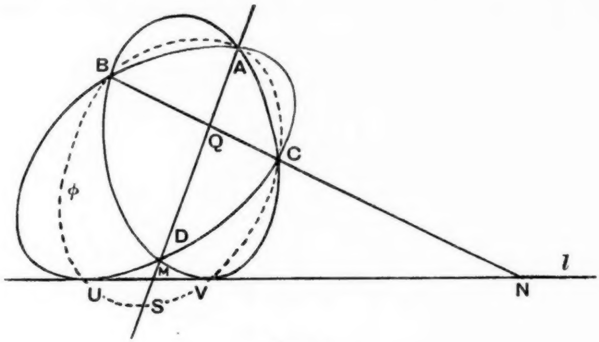

FIG. 2.

Theorem B. Let A, B, C and D be the base points of a pencil of conics. This pencil determines an involution on the line l. Consider the conic ϕ passing through A, B, C and the double points U and V of the involution. Then the cross-ratio $\{D, S, Q, M\} = -1$, where $Q = (BC, AD)$, $M = (AD, l)$ and S is the second point of the conic on AD.

Although, according to our principle, the theorem is already proved, it may be desirable to give another proof on more orthodox lines. The following will be found to proceed in close analogy with the original one.

Denote (BC, l) by N. We have

$$A\{C, S, U, V\} = B\{C, S, U, V\} \text{ (on conic } ABCSUV). \quad\text{...........(1)}$$

AD, CB and AC, BD are corresponding rays in an involution. This latter generates a point involution on line l having the double points U and V.

Hence $$A\{C, S, U, V\} = B\{D, C, U, V\}. \quad\text{.........................(2)}$$

It follows from (1) and (2) that

$$B\{C, S, U, V\} = B\{D, C, U, V\} = B\{C, D, V, U\}. \quad\text{................(3)}$$

Moreover, upon l,

$$\{M, N, U, V\} = -1, \text{ hence } B\{M, C, U, V\} = -1. \quad\text{......................(4)}$$

We define now two concentric projective ranges by giving the following pairs of corresponding elements:

$$(BU, BV), \quad (BV, BU) \quad \text{and} \quad (BC, BC).$$

This is seen to be an involution, with BC as one of its double rays. Because of (4), BM will be its other double element.

Because of (3), BS, BD are also corresponding elements, and thus we have

$$B\{D, S, C, M\} = B\{D, S, Q, M\} = -1. \qquad \text{Q.E.D.}$$

From the projective Theorem B metric facts can be deduced by taking any two points of the figure to be the circular points. If this is done in respect of the double points of the involution, once more Theorem A is obtained. But if B and C play this part, then BC is necessarily the line at infinity and Q the point at infinity of AD. The double points U and V of the involution determined upon l can be obtained by constructing those circles in the pencil AD, which touch the line l. The conic through $ABCUV$ becomes the circle through A, U and V. Let its point of intersection with AD be called S, as before. We obtain then (Fig. 3):

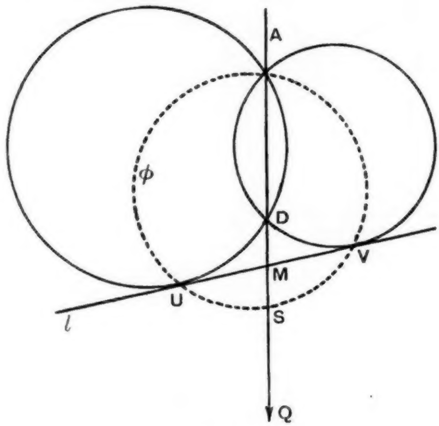

FIG. 3.

Theorem C. Two circles intersect in A and D. Draw one of the common tangents l which touches at U and V respectively. Draw further the circle ϕ through A, U and V. If $M=(AD, l)$ and S is the second point of AD on the circle ϕ, then $DM=MS$.

Considered from a Euclidean point of view this theorem has no connection whatsoever with Theorem A. It can, of course, also be proved in an elementary way:

$$MU^2=MV^2=MD \,.\, MA.$$

Also $$MV \,.\, MU=MS \,.\, MA.$$

Hence $MS=MD$.

We could now find other theorems by specialising other points occurring in Theorem B. However, we mention only the case when the conics degenerate into pairs of straight lines (Fig. 4).

Let A, B, C, D be the vertices of a square, AD a diagonal, l be again the line at infinity. Then the conics through the four points, which touch l, are two line pairs, viz. AB, CD and AC, BD. They touch in the points at infinity AB (or CD) and AC (or BD) respectively. The conic through these two points and through A, B and C again degenerates into AB and AC. Thus M is the point at infinity of AD, Q is the centre of the square and $S=A$. $\{D, S, Q, M\}=-1$ means that the diagonals bisect each other.

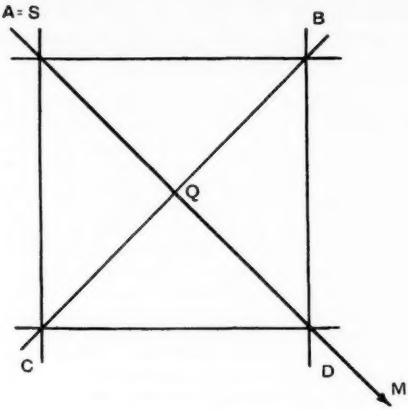

FIG. 4.

This theorem is, in its turn, also a special case of yet another projective fact, concerning the points forming a harmonic range on the diagonal of a complete quadrilateral. S. V.

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ON THE BEHAVIOUR OF THE ROOTS OF AN ALGEBRAIC EQUATION AS THE COEFFICIENTS VARY.

By J. C. Jaeger.

There are many methods of solving algebraic equations whose coefficients are real and have given numerical values. In practice it very often happens that the coefficients are not given numerically, but are expressed in terms of certain parameters, and that what is wanted is not the roots of the equation for special values of the parameters, but an indication of the way in which the roots behave as these parameters vary between certain limits. For example, if the equation is the period equation of a mechanical or electrical system, we may wish to know how its roots vary as certain components of the system are changed, not merely the values of the roots for definite values of the components. The method sketched below has been found very useful for studying cubics from this point of view and applies quite well to quartics. It also provides interesting elementary examples in the theory of equations.

Since we are not concerned with particular values of the coefficients, we regard the roots as the fundamental quantities and determine the coefficients in terms of them. As a first example, consider the quadratic

$$x^2 + 2kx + \omega^2 = 0, \qquad \text{(1)}$$

whose roots are $-k \pm \sqrt{(k^2 - \omega^2)}$. From the present point of view this is equivalent to the statement that if $\xi^2 < \omega^2$ and $\eta = \sqrt{(\omega^2 - \xi^2)}$, equation (1) with $k = \xi$ has complex roots $-\xi \pm i\eta$; while, if $\xi^2 > \omega^2$, equation (1) with $k = \xi$ has a pair of real roots, half whose sum is $-\xi$ and half whose difference is $\sqrt{(\xi^2 - \omega^2)}$. The complete behaviour of the roots of (1) for all values of k and ω can now be represented in the diagram of Fig. 1 (only the portion $k > 0$ is shown, that for $k < 0$ is symmetrical about the vertical axis) in which, if there is a complex root $-\xi + i\eta$, its position is plotted, while, if there is a pair of real roots, half their sum is plotted as abscissa, and half their difference as ordinate, measured downwards to avoid confusion with the part of the diagram for complex roots. As k in (1) is increased from zero, the roots of (1) are determined by a point which starts from A and moves along the curve. For example, at the point P, (1) with $k = \frac{1}{2}\omega$ has roots $-\frac{1}{2}\omega \pm \frac{1}{2}\omega\sqrt{3}$; while, at the point Q, (1) with $k = \frac{3}{2}\omega$ has a pair of real roots $-\frac{3}{2}\omega \pm \frac{1}{2}\omega\sqrt{5}$. The dotted curve in Fig. 1 is the corresponding diagram for the quadratic

$$x^2 + 2kx - \omega^2 = 0.$$

Thus the roots of any quadratic follow immediately from the curves of Fig. 1.

A cubic equation is best treated in the form in which it arises in practice, since then the parameters occurring in the equation will usually have a simple and important physical significance which would be obscured if other parameters were introduced or the equation transformed in any way. For this reason a particular example is treated below, though the same method can be applied to any case. Consider the equation

$$(x^2 + 1)(x + c) + kx^2 = 0, \qquad \text{(2)}$$

where k and c may have any positive values.* Suppose the roots of (2) are $-\alpha$ and $-\xi \pm i\eta$, then, writing

$$r^2 = \xi^2 + \eta^2, \qquad \text{(3)}$$

* This restriction keeps the real parts of the roots negative, and this is the case of practical importance, but the method applies to other values of k and c. The equation is the period equation of a feedback servomechanism; c depends on the circuit components, and k on the amount of feedback.

the relations between roots and coefficients give

$$\alpha + 2\xi = c + k, \quad \text{......(4)}$$

$$2\xi\alpha + r^2 = 1, \quad \text{......(5)}$$

$$\alpha r^2 = c. \quad \text{......(6)}$$

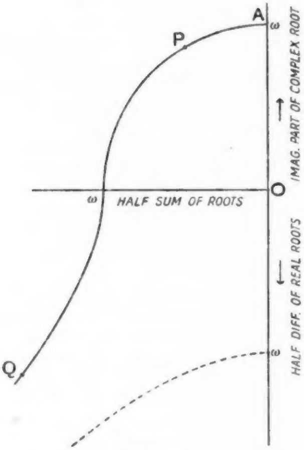


Fig. 1.

Now suppose we regard α and c as fundamental and express the other quantities in terms of them.

Then we have

$$r^2 = c/\alpha, \quad \text{......(7)}$$

$$\xi = \tfrac{1}{2}(\alpha - c)/\alpha^2, \quad \text{......(8)}$$

$$k = \alpha + 2\xi - c = (\alpha - c)(1 + \alpha^{-2}), \quad \text{......(9)}$$

$$\eta = \surd(r^2 - \xi^2). \quad \text{......(10)}$$

That is, for the value of c under consideration, $-\alpha$ and $-\xi \pm i\eta$ given by (7), (8) and (10) are the roots of (2) with k given by (9). If, for some values of α, η given by (10) turns out to be imaginary, say $i\delta$, a repetition of the same argument assuming three real roots shows that (2) with k given by (9) has three real roots $-\alpha$ and $-\xi \pm \delta$.

The positions of the roots of (2) for all values of k and c may now be shown on a single figure; for each value of c the position of the complex root $-\xi + i\eta$ is plotted for values of α increasing from c to ∞; if for any value of α there are two other real roots, the mean of these, $-\xi$, is plotted as abscissa, and half their difference, δ, as ordinate, measured downwards. Then for each value of c there is a continuous curve, corresponding to α increasing from c to ∞, and this shows immediately the regions in which there are three real roots. A family of such curves for various values of c is shown by the full

lines in Fig. 2. The points on these curves which have the same values of α/c are joined by the dotted curves, and the solution can be read off immediately since k follows from (9), that is, $k = \alpha + 2\xi - c$. For example, on the curve $c = 1/6$ in Fig. 2, at the point P we find that $-\cdot 25$ and $-\cdot 6667 \pm \cdot 4714i$ are the roots of (2) with $k = 1\cdot 4167$. Similarly at the point Q, $-1/3$ and $(-3 \pm 1)/4$ are the roots of (2) with $k = 5/3$.

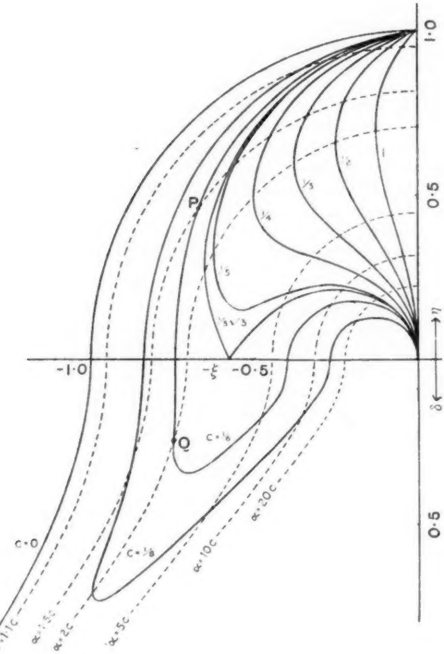


FIG. 2.

Fig. 2 shows immediately that, if $c > 1/3\sqrt{3}$, there is only one real root, while, if $c < 1/3\sqrt{3}$, there is a pair of complex roots for small, and for large, values of α (and of k) and three real roots for intermediate values. Also it shows the way in which the roots of (1) behave for large and small values of c. The curve marked $c = 0$ is the corresponding diagram for the quadratic $x^2 + kx + 1 = 0$ found in Fig. 1, while as $c \to \infty$ the curves tend to the imaginary axis.

As remarked above, a diagram of this type can be constructed for any cubic. In applying the method to quartics it is often known from physical considerations which pair of roots is of the greatest interest, and the variation of these, when the product of the other roots is taken as parameter, can be studied.

J. C. JAEGER.

MOTION OF A PARTICLE.

BY N. M. H. LIGHTFOOT.

SUPPOSE that a particle of mass m is attached to a fixed point O, by means of a light inextensible string of length a, and when hanging at rest is suddenly given a velocity u_0 at right angles to the string. The earlier phases of the motion are dealt with in standard textbooks, but I do not recall having seen a complete analysis of the later stages in the case when the particle leaves its circular path. It is well known that if $u_0^2 \leqslant 2ag$, the particle executes oscillations like a simple pendulum (this will be referred to later as the case of simple oscillations), while if $u_0^2 \geqslant 5ag$, it makes complete revolutions in a circle about O. If $2ag < u_0^2 < 5ag$, the particle leaves its circular path and then travels in a parabolic path until the string becomes taut again. The textbooks do not follow the motion beyond this point, perhaps because it is realised that no real string fulfils the condition of being inextensible. If, however, we assume that the ideal string exists, the resulting motion has some rather surprising features. Owing to the obvious loss of energy, the particle when it first passes through the lowest point again will have a smaller velocity that it had initially, and so will either execute simple oscillations or will leave its circular path at a lower point than previously and lose further energy when the string tightens a second time. It would at first appear that ultimately the energy must decrease to such a value that the particle will finally be moving with simple oscillations. This is not always true, as the following analysis will show.

Let $u_1, u_2, u_3, \ldots$ be the successive velocities with which the particle returns to its lowest point and write $\lambda_0 = u_0^2/ag$, $\lambda_1 = u_1^2/ag$, etc. Then if $\lambda_0 \leqslant 2$ or if $\lambda_0 \geqslant 5$, we have $\lambda_0 = \lambda_1 = \lambda_2 = \ldots$. Thus we may restrict our investigation to the case of $2 < \lambda_0 < 5$. Let u be the velocity of the particle at a point in its circular path at which the string is inclined at an angle θ to the upward vertical, T the tension in the string at this point, and $\lambda = u^2/ag$. Then elementary considerations give as usual :

$$u^2 = u_0^2 - 2ag(1 + \cos\theta),$$

$$T + mg\cos\theta = mu^2/a,$$

or

$$\lambda = \lambda_0 - 2(1 + \cos\theta),$$

$$\cos\theta + T/mg = \lambda.$$

The particle leaves the circle when $T = 0$, that is, where

$$\lambda = \cos\theta = (\lambda_0 - 2)/3.$$

At this point let $\theta = \alpha$, $u = u'$ and $\lambda = \lambda'$. The coordinates of this point referred to horizontal and vertical axes through O, the positive direction of the x-axis being that of the initial velocity, are $(a\sin\alpha, a\cos\alpha)$. The particle then begins to move in a parabolic path with initial velocity u' at an angle α to the horizontal. Its coordinates at a subsequent time t are thus given by

$$x = a\sin\alpha - u'\cos\alpha \, . \, t,$$

$$y = a\cos\alpha + u'\sin\alpha \, . \, t - \tfrac{1}{2}gt^2.$$

The coordinates of the point where the particle reaches the circle again and the time of this occurrence are given by solving these equations together with the equation of the circle, $x^2 + y^2 = a^2$, for x, y and t. Thus we obtain

$$t = 4u'\sin\alpha/g,$$

$$x = a\sin\alpha\,(1 - 4\cos^2\alpha),$$

$$y = a\cos\alpha\,(1 - 4\sin^2\alpha).$$

If β is the angle between the radius to this point and the upward vertical, measured in the same sense as α, then

$$\sin\beta = \sin\alpha\,(1 - 4\cos^2\alpha), \qquad \cos\beta = \cos\alpha\,(1 - 4\sin^2\alpha).$$

The radial momentum is destroyed, but the tangential momentum is unaffected. The tangential velocity is thus $\dot{y}\sin\beta - \dot{x}\cos\beta$, which for the above value of t reduces to $\sqrt{ag\lambda'}(-3 + 12\lambda'^2 - 8\lambda'^4)$. Hence using the equation of energy, we have

$$u_1^2 = ag\lambda'(-3 + 12\lambda'^2 - 8\lambda'^4)^2 + 2g(a + y),$$

where y has the value found above. This gives after some reduction

$$\begin{aligned}\lambda_1 &= \lambda_0 - 64\lambda'^3(1 - \lambda'^2)^3\\ &= \lambda_0 - \frac{2^6}{3^9}(\lambda_0 - 2)^3(\lambda_0 + 1)^3(5 - \lambda_0)^3.\end{aligned}$$

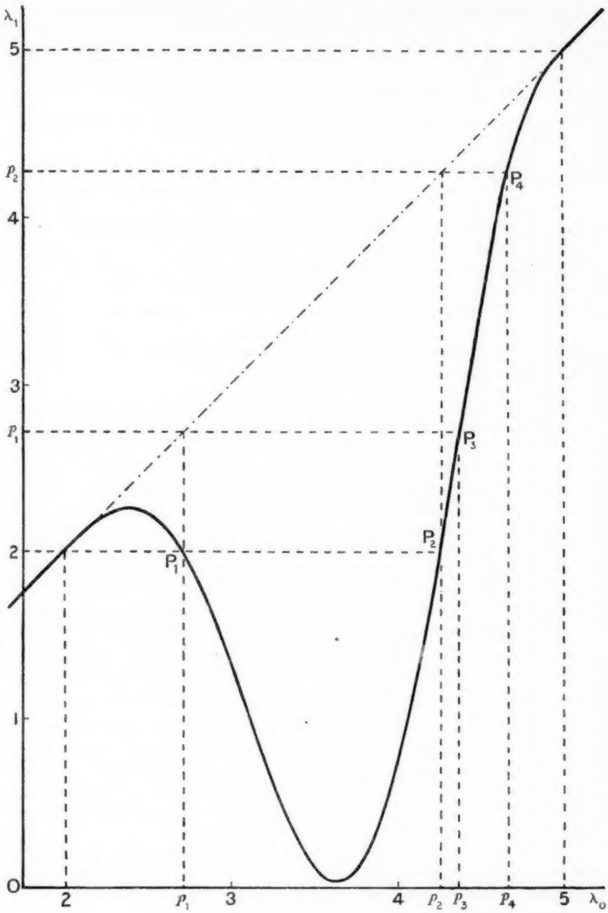
As was to be expected, this equation shows that $\lambda_1 = \lambda_0$, when $\lambda_0 = 2$ or $\lambda_0 = 5$. The value of λ_2 (if $\lambda_1 > 2$) is obviously given by replacing λ_0 by λ_1 in the right side of this equation, and so on. The complete relationship between λ_1 and λ_0 is shown graphically in the figure, the graph being a straight line for values of λ_0 less than 2 or greater than 5. The coordinates of the turning points are not of great importance in what follows, but they have been determined approximately as $\lambda_0 = 2{\cdot}392$, $\lambda_1 = 2{\cdot}256$, and $\lambda_0 = 3{\cdot}637$, $\lambda_1 = 0{\cdot}03591$.

The points which are of immediate interest are P_1 and P_2 where $\lambda_1 = 2$. If we denote the values of λ_0 at these points by p_1 and p_2 respectively, it is clear that if $2 < \lambda_0 < p_1$, then the energy remaining at the first return to the lowest point of the circle is sufficient to carry the particle once more above the level of the centre, and as it is also evident that $2 < \lambda_1 < p_1$, the particle will again return to the lowest point with sufficient energy to rise again above the centre, and so on indefinitely. The energy of course is reduced at each return, and the path described may be regarded as a circular arc, rather greater than a semicircle, with a small parabolic loop at each end, the loops getting smaller and smaller as time goes on, and the motion tending as t tends to infinity to a true semicircular motion. The motion is never, in finite time, a simple oscillation.

If, on the other hand, $p_1 < \lambda_0 < p_2$, then $\lambda_1 < 2$, and after one deviation from circular motion the particle executes simple oscillations, with amplitude dependent on the value of λ_0, being smallest when $\lambda_0 = 3{\cdot}637$.

In order to discuss the case of $\lambda_0 > p_2$, let the points $P_3, P_4, P_5, P_6, \ldots$ be taken on the graph with abscissae $p_3, p_4, p_5, p_6, \ldots$, the ordinates of these points being $p_1, p_2, p_3, p_4, \ldots$ respectively. Only P_3 and P_4 are shown in the figure, but the method of construction is obvious. Clearly if $p_2 < \lambda_0 < p_3$, then $2 < \lambda_1 < p_1$, and therefore $2 < \lambda_2 < p_1$, and so on, so that the motion is ultimately similar to the first type described above; while if $p_3 < \lambda_0 < p_4$, then $p_1 < \lambda_1 < p_2$ and $\lambda_2 < 2$, so that after two deviations from the circular path the particle executes simple oscillations never again rising above the centre. The process can obviously be extended and in general if $p_{2n} < \lambda_0 < p_{2n+1}$ the particle always rises above the centre after passing through the lowest point, and ultimately does so at each extremity of its path, as described in the first case, while if $p_{2n-1} < \lambda_0 < p_{2n}$, the particle describes n parabolic arcs and then settles down to simple oscillations. If $\lambda_0 = p_{2n-1}$ or $\lambda_0 = p_{2n}$, then after n parabolic arcs have been described the motion becomes exactly semicircular.

The numbers $p_1, p_2, p_3, \ldots$, obviously form an increasing sequence which has 5 as its limit. Approximate values have been determined by an iterative process which need not be described here, and the first few are shown in the table (opposite).



p_1	p_2	p_3	p_4	p_5	p_6	p_7
2·7079	4·2519	4·37	4·67	4·70	4·79	4·81

It is obvious that the difference between successive values soon becomes very small, and that as λ_0 approaches 5 a very small change in initial velocity may lead to a considerable difference in the ultimate form of the motion. This result was not expected when the investigation was undertaken.

N. M. H. L.

THE LOGARITHMIC FUNCTION; AND THE NUMBERS e AND π.

By D. K. Picken.

I. The Logarithmic Function and the Number e.

1. This perennial question has come up, once again, in a Note (No. 1805, February 1945) by the President for the year.

The object of this contribution is to try to state something like a full case (the writer seems to have been working at this for most of his academic life) for the traditional view of the subject, for which the President contends: while at the same time indicating the complementary importance of the other view. (As in all such cases, strong difference of view by competent authorities implies important truth on both sides (which it is necessary to synthesise; and this is, in fact, a fundamental mathematical case of the kind).)

Much of the ground has been covered by the writer in the pages of the *Gazette* (and elsewhere), and need, therefore, only be treated by specific references and by summary; but there is a rounding up of essential ideas which is the main purpose of this contribution.

2. The first point to be emphasised is that *logarithm* is, of course, quite an elementary conception—in the context of the Natural Numbers—bound up with *power* and *root*, and correlated with *sum* and *difference*, and with *product* and *quotient*, in the seven basic number operations.* And, in that context, all the fundamental logarithm-propositions are also elementary †—though not of very obvious importance, because of the extreme restriction to which they are there subject. That they are, nevertheless, of very real importance is evident in the fact that, but for elementary grasp of the *forms* of the propositions in question, the whole business of the use of logarithm tables—involving propositions of these same forms, which are incomparably more difficult to prove—would be ever so much more of a practical problem (to an extent we can, perhaps, hardly realise). These are forms we have to *know* long before we can understand (if, indeed, we ever can ‡) the proofs for the general " real " case—which is, in fact, the practically important one. And something closely analogous is true in the relevant Calculus theory.

3. The second point needing emphasis is that, in the general Real Number case, the one-valued $\log_a b$ (where a, b are both positive) is logically prior to that of a^b (where a is positive). This fact asserts itself in the " antilogarithms " of the tables, and in the calculation of powers by means of logarithms; also in the process of what is sometimes called " logarithmic differentiation " (of functional power-forms).

Further, from the Dedekind definition (which is, in fact, fundamental), of $\log_a b$ and a^b, in the general real case, it follows that the two mutually inverse functions specified by $\log_a x$ and a^x (where $a(\neq 1)$ is a given positive base) are both monotonic continuous functions in the respective relevant real ranges of x; and monotonic *increasing* if $a > 1$.§

* See *Gazette*, XXII, 250, pp. 226–8 (July 1938); and *The Number-System of Arithmetic and Algebra* (Melbourne University Press), by the writer (in the M.A. Library).

† Such a secondary form as $\log_b a = 1/\log_a b$ is *not* elementary, because only one of the two logarithms in question is there admissible.

‡ Most of the workers who use logarithm tables freely have no full knowledge of the difficult theory involved.

§ The variation of these functions is discussed in Appendix IV of the above-mentioned *Number-System* (one simple step being inadvertently omitted there).

Using (say) $a=2$, the graphs of these two functions can then be drawn, by means of a few easily "plotted" points—and should be familiar facts: the respective "concavity downwards" and "concavity upwards" being an immediate consequence of the propositions

$$\log_a (x+h) - \log_a x = \log_a (1+h/x)$$

and

$$a^{x+h} - a^x = a^x(a^h - 1),$$

the former of these differences decreasing, the latter increasing, as x increases —for any *given* increment h.*

4. For the systematic process of Differentiation, the simplest procedure would appear to be to build up from one or two basic cases (primarily $y=x$ and y *constant*) by means of the general theorems on the differentiation of (i) $u \pm v \pm \dots$; (ii) $u \overset{\times}{\div} v \overset{\times}{\div} \dots$; (iii) function of a function, and the inverse of a function; (iv) $\log_u v$ and u^v—where $u, v, \dots$ denote functions which can be differentiated.

The two forms (iv) reduce (by means of (ii) and (iii)) to the two basic forms $\log_a x$, a^x—where a is an auxiliary constant base; and for differentiation of these we may proceed in either order (with or without use of the theorem on the inverse function); actually, as throughout Real Number theory, it is better to proceed from the logarithm form.†

If $y = \log_a x$, $\delta y = \log_a (1+\delta x/x) = \log_a (1+\xi)$, say; and

$$\delta y/\delta x = \{\log_a(1+\xi)\}/\xi x = \{\log_a(1+\xi)^{1/\xi}\}/x,$$

where $\xi (= \delta x/x) \to 0$ with δx (since $x > 0$).

The problem of differentiating these two forms thus reduces to investigation of the limiting value (if any) of $(1+x)^{1/x}$ when $x \to 0$.

The steps, so far, are straightforward and (as steps) surprisingly simple. The difficulty of proof of the two logarithm-theorems involved (on quotient and power, in the general real case) is precisely the type of difficulty that is implicit in all the important practical uses of logarithms.

5. Writing $y = (1+x)^{1/x} = a^u$, using an auxiliary constant real base a (greater than 1),

$$u = v/x, \text{ where } v = \log_a(1+x).$$

We can then work back from the known monotonic increasing continuous variation of v, as x increases from -1 to $+\infty$—shown in a graph by the concave downwards locus of a point V (drawn for $a=2$, say).

The function u, being the gradient of the radial line OV, has monotonic decreasing continuous variation:

from $x = -1$ (where $u = +\infty$) to $x = 0-$

and from $x = 0+$ to $x = +\infty$ (where $u = 0$)‡

—the question of value(s) for u at $x=0$ being the limit-problem under investigation.

And so, from the known monotonic continuous variation of a^x, it follows

* It is notable that in both cases the variation of the "difference" is expressible in terms of that of the function itself. This is characteristic simplicity of the fundamental.

† See § 3, above; also *Gazette*, XIII, 190, pp. 411-3 (October 1927).

‡ It is easy to reduce the graphical discussion of the gradient-variation to analytical form (in terms of the coordinates of three points V_1, V_2, V_3), and—on the basis of the monotonic variation—to argue that $u \to 0$ when $x \to \infty$: it being sufficient to use $1+x=a^n$, with n integral and tending to ∞.

that $y(=a^u)$ has monotonic decreasing continuous variation :

$$\text{from } x=-1 \text{ (where } y=+\infty) \text{ to } x=0-$$
$$\text{and} \quad \text{from } x=0+ \text{ to } x=+\infty \text{ (where } y=1).$$

Thus simple graphs for u and for y can be drawn, for $x<0$ and $x>0$. (These are so simple that they are here omitted, to save space.)

6. The simplicity of these facts—of monotonic continuous variation—paves the way for the final steps of the investigation.

The expression $(1+x)^{1/x}$, in the real range $(-1, +1)$ of x, includes the two sequence forms $(1\pm 1/n)^{\pm n}$, for $x=\pm 1/n$ and n integral. And elementary Binomial expansion gives

$$(1+1/n)^n=1+1+\sum_{r=2}^{r=n}\{1-1/n\}\{1-2/n\}\ldots\{1-(r-1)/n\}/r!$$
$$=1+1+\Sigma(1-p_r)/r!, \text{ say ; and } 0<p_r<1.$$

It increases with n (obviously, or by § 5 above) but is less than

$$1+\sum_{r=1}^{r=n}1/r!<1+\sum_{1}^{\infty}1/r!$$

($<e$, so denoting the sum of this simple convergent infinite series).

Also
$$p_r<\{1+2+\ldots+(r-1)\}/n,$$
$$<r(r-1)/2n^*.$$

Hence
$$(1+\sum_{1}^{n}1/r!)-(1+1/n)^n<\frac{1}{2n}\{1+\sum_{r=3}^{r=n}1/(r-2)!\}$$
$$<\frac{1}{2n}\left\{1+\sum_{r=1}^{r=n-2}1/r!\right\}<e/2n\,;\dagger$$

and $(1+1/n)^n\to e$, when $n\to\infty$.

Also $(1-1/n)^{-n}=\{n/(n-1)\}^n=(1+1/N)^{N+1}\to e$, $(n=N+1)$; and it therefore follows—from the nature of the continuous variation of $y=(1+x)^{1/x}$, as determined above—that $(1+x)^{1/x}\to e$, when $x\to 0\pm$; the graph of y being completed—continuous—by the point $(0, e)$.

7. Thus, returning to § 4 above,

$$\text{if} \quad y=\log_a x, \quad y'=(\log_a e)/x\,;$$

and so, again, if $y=\log_e x$, $y'=1/x$,‡ where e denotes the sum of the convergent infinite series

$$1+1+1/2!+1/3!+\ldots.$$

It is this simple result that establishes e as the *theoretical* " natural " base for logarithms—so that, when we write $y=\log x$, the base e is implied.

8. There is fascination in the complete discussion of this fundamental question—for those susceptible to such fascination (by no means confined to

* $r(r-1)<2n$, for values of r up to a certain value $k(<n)$; when $r>k$, the inequality for p_r is implied by $0<p_r<1$ (*e.g.* if $n=15$, $k=5$).

† Upper limit in terms of e itself : another remarkable instance of simplicity of the fundamental.

‡ The particular—simpler—result being remembered, the general result can be " recovered " by using $\log_a x=\log x/\log a$.

the essentially mathematical): especially as it leads on immediately to the remarkable generalisation of the central proposition of § 6, viz. that the sequence specified by $(1+z/n)^n$ tends to the limit $E(z)$ determined as the sum of the convergent infinite series $1+\Sigma z^r/r!$—for all values of z (real or unreal).

The proof is an obvious generalisation of that in § 6: thus

$$\left|\left(1+\sum_{r=1}^{r=n}\frac{z^r}{r!}\right)-\left(1+\frac{z}{n}\right)^n\right|=\left|\sum_{r=2}^{r=n}(p_r\,.\,z^r/r!)\right|$$
$$<\sum_{r=2}^{r=n}(p_r\,.\,\rho^r/r!),\quad \text{if } \rho=|z|,$$
$$<\rho^2E(\rho)/2n,\ \text{as in § 6}$$
$$\to 0,\quad \text{when } n\to\infty.$$

9. But the discussion can hardly be regarded as elementary, and is therefore unsuitable for a first approach to the question—except in one respect, viz. that the essential *facts* of the argument of §§ 4–6 can be very simply summarised (in a few lines) because of elementary knowledge of the *forms* of the relevant logarithm-propositions.

It is therefore suggested that the President's point can be met by giving some such summary of the direct process of differentiating $\log_a x$ and the emergence of the number e; and then pointing out that this is, in fact, a transition case, which can be approached the other way round, viz. as emerging from the exceptional case * of the "anti-differentiation" of x^n (for n integral) when $n=-1$.

This makes of it the simplest case of the specification of functions in a new way, viz. by means of a definite integral expression—or by means of a differential equation (here the simple case of $xDy=1$). This has long been used as giving elementary approach to the infinite expansion of $\log(1+x)$, from that of $1/(1+x)$.

II. The Circular Functions and the Number π.

10. (i) There are certain points of analogy in the one other standard case of elementary differentiation, viz. that of the Trigonometrical Functions—which are defined in terms, not of number-operations, but of the geometry of Similar Triangles.

Here, in $\sin x$, etc.,—where x must (for purposes of differentiation) be a *number*-variable—there is again an initial arbitrary element (like the constant a in $\log_a x$ and a^x), viz. the unit of angle, by reference to which the *number*-argument, x, specifies the *angle*-argument of the trigonometric function.

(ii) The fundamental limit-problem of this case (viz. that of $\sin x/x$ when $x\to 0$) is solved by the elementary geometrical inequality which is expressible in the trigonometric form

$$\sin X<(\text{circular measure of } \angle X)<\tan X,\ \text{for acute } X,$$

and thence
$$\sin x/x<k<\tan x/x,$$

where k denotes the circular measure of the arbitrary unit angle (in terms of which x is the measure of X); whence, again,

$$k\cos x<\sin x/x<k<\tan x/x<k\sec x,$$

from which it follows that $\sin x/x\to k$ and $\tan x/x\to k$, when $x\to 0$.† And for

* It is worth noting that "exceptional cases" in mathematics are practically always of positive significance, and are commonly gateways to important developments.

† This more general form of these limits has commonly been overlooked (see, *e.g.* the present writer, *Gazette*, XIII, 190, p. 414), and the corresponding differentiations deduced from the standard special case.

the differentiations, we have

$$D \sin x = k \,.\, \cos x,\ D \cos x = -k \,.\, \sin x,\ D \tan x = k \,.\, \sec^2 x, \text{ etc. };$$

and, therefore, $(D^2 + k^2)y = 0$, if $y = \sin x$ or $\cos x$.

When $k = 1$, we get the standard simple forms—*i.e.* when $\sin x$, etc., *mean the sine, etc., of the angle whose circular measure is* x. And, when so defined, $y = \sin x$, etc., are called " the Circular Functions ". (These functions can, of course, also be taken as an elementary case for investigation from a simple differential equation, $(D^2 + 1)y = 0$.)

(iii) The unit angle of circular measure (*i.e. the radian*—properly so defined) is the analogue, in this case, of the base e for the logarithmic function. And it is specified by means of the number π, which is the circular measure of the straight angle (S, say) : so that

$$S = \pi \,.\, \text{radian} \quad \text{and} \quad \text{radian} = S/\pi = (180/\pi)^\circ.$$

The circular functions are periodic functions, of which π is the half-period; and $\sin \pi = 0$, $\cos \pi = -1$, $\sin \frac{1}{2}\pi = 1$, $\cos \frac{1}{2}\pi = 0$; $\tan \frac{1}{4}\pi = 1$: the last of these leading to the algebraic form

$$\tfrac{1}{4}\pi (= \text{arc} \tan 1) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots \textit{ ad inf.},$$

by means of which (directly or indirectly) the decimal expression for π can be calculated to any required degree of accuracy.

11. (i) The " decimal " expressions for e and π, when the radix of numeration is (1) radix 8, thence (2) radix 2, are worth noting.* These are :

(1) radix 8, $e = 2{\cdot}55760\ 52310\ 50535\ 51246 \ldots,$

$\pi = 3{\cdot}11037\ 55242\ 10264\ 30215 \ldots.$

(2) so, with radix 2,

$e = 10{\cdot}10110\ 11111\ 10000\ 10101\ 00110\ 01000\ 101 \ldots,$

$\pi = 11{\cdot}00100\ 10000\ 11111\ 10110\ 10101\ 00010\ 001 \ldots.$

These latter are the basic forms : 0 and 1 being absolute in the number-system.

(ii) Nothing could be more eloquent of the joint place of these two numbers in the fundamental structure of Mathematics than the relation :

$$(e^{i\pi})(= E(i\pi) = \cos \pi + i \sin \pi) = -1,\dagger$$

of which Klein has said : " This formula is certainly one of the most remarkable in all mathematics." D. K. P.

* See *Gazette*, XXII, 250, pp. 229, 233, for the inter-relation of these two cases, and of the following " decimal " forms—which were taken out by modern calculating machines (used in Munitions work) by Mr. J. A. Macdonald, M.A., a former student of the writer.

† $(e^{i\pi})$ for " principal value " : the general " power " form being $a^b = E(b \,.\, L(a))$, where $L(z)$ is the " inverse " of $E(z)$, and is equal to $\log |z| + i \,.\, \text{Amp}\,(z)$.

GLEANINGS FAR AND NEAR.

1491. An excellent form of waterproof coat . . . is a large waterproof sheet made after the fashion of a poncho. This consists of a square sheet of mackintosh, measuring say 8 feet by 5 feet, with a hole in the centre large enough for the head to pass through.—*Mountaineering* (Badminton Library), Chap. XXX, 2. [Per Mr. W. A. Thompson.]

ON THE CONSTRAINED MOTION OF A RIGID BODY.

BY C. JAYARATNAM ELIEZER.

1. The idea of the "reversed effective force" or "centrifugal force" has been widely used to facilitate the solution of certain types of dynamical problems concerning the motion of particles. But the method can also be applied, as will be shown below, to investigate the constrained motion of a rigid body.

Suppose the point O of a rigid body is constrained to move in some known way, while the body is acted upon by certain given external forces $\mathbf{F}$. We shall consider the case when the constraint is smooth. Let us denote the position vector of the point O with respect to a fixed frame of reference S by $\mathbf{r}_0$, its velocity by $\mathbf{v}_0$, and its acceleration by $\mathbf{f}_0$. We divide the rigid body into small elements and suppose that a typical element has position vector $\mathbf{r}_1$ with respect to a set of moving axes S_1, which pass through O and which are parallel to the axes of the fixed frame of reference S. Let the force of constraint at O be $\mathbf{F}_0$.

The equation of motion of the element of mass m is

$$\mathbf{X}+\mathbf{X}'=m\mathbf{f}=m(\mathbf{f}_0+\mathbf{f}_1) \qquad \text{.................................(1)}$$

where the suffix 1 indicates that $\mathbf{f}_1$ is relative to the moving axes S_1, $\mathbf{X}$ refers to the external forces and $\mathbf{X}'$ to the internal forces acting on the element. When summing up the equations of motion for all such elements, the internal forces cancel out, and we obtain the equations of motion of the body.

2. *The equations of motion.*

(i) *Equations of translatory motion.* From the equation (1), by adding the equations of motion of all the elements of the body, we obtain

$$\mathbf{F}+\mathbf{F}_0=\Sigma m(\mathbf{f}_0+\mathbf{f}_1)=M\mathbf{f}_0+M\bar{\mathbf{f}}_1, \qquad \text{............................(2)}$$

where $\bar{\mathbf{f}}_1$ denotes the acceleration, relative to the moving axes S_1, of the centre of inertia G of the body. The position vector and the velocity vector of G with respect to S_1 will be denoted by $\bar{\mathbf{r}}_1$ and $\bar{\mathbf{v}}_1$ respectively.

(ii) *Equation of moments.* If we take moments about any point whose position vector with respect to S_1 is $\mathbf{x}_1$, then we have

$$\begin{aligned}\mathbf{M}+\mathbf{M}_0&=\Sigma m(\mathbf{r}_1-\mathbf{x}_1)\wedge(\mathbf{f}_0+\mathbf{f}_1)\\&=\Sigma m(\mathbf{r}_1-\mathbf{x}_1)\wedge\mathbf{f}_1+M(\bar{\mathbf{r}}_1-\mathbf{x}_1)\wedge\mathbf{f}_0, \qquad \text{............................(3)}\end{aligned}$$

where $\mathbf{M}$ and $\mathbf{M}_0$ are the moments about $\mathbf{x}_1$ of $\mathbf{F}$ and $\mathbf{F}_0$ respectively.

3. We can write the equations (2) and (3) in the form

$$\mathbf{F}+\mathbf{F}_0+\mathbf{R}=M\bar{\mathbf{f}}_1, \qquad \text{..(4)}$$

$$\mathbf{M}+\mathbf{M}_0+(\mathbf{r}_1-\mathbf{x}_1)\wedge\mathbf{R}=\Sigma m(\mathbf{r}_1-\mathbf{x}_1)\wedge\mathbf{f}_1 \qquad \text{......................(5)}$$

where $\mathbf{R}=-M\mathbf{f}_0$. Hence we see that the equations which determine the relative motion are the same as when the point O is at rest, and an additional force $-M\mathbf{f}_0$ acts at the centre of inertia G. Thus we derive the following method of solution: *the relative motion is obtainable by supposing that O is at rest and introducing an additional force $-M\mathbf{f}_0$ at the centre of inertia.*

We may regard this additional force either as a force $-M\mathbf{f}_0$ acting at the centre of inertia or as a field of force of strength $-\mathbf{f}_0$ per unit mass.

There are two possible ways of viewing the problem. One is to regard the additional force so introduced, that is, the centrifugal force, as a "*fictitious*" force.* This force acts on the body when the moving constraint is "*ficti-*

* Whittaker, *Analytical Dynamics*, Cambridge (1937), p. 40.

tiously " brought to rest, and leads to the same equations of motion as in the given problem.

The other possibility is to describe the motion from the point of view of an observer who is moving with the constrained point O. The fictitiousness then disappears. The forces involved are then all real. The forces which this moving observer reckons may be called the " apparent forces ".* These apparent forces will be equivalent to the sum of the given external forces $\mathbf{F}$ and the centrifugal force $-M\mathbf{f}_0$. When the external forces on the body include its weight, that is, the motion is under gravity, we may include the centrifugal field of force of strength $-\mathbf{f}_0$ in the gravitational field, and so obtain a modified gravitational field, which may be called the apparent gravitational field. Its strength is

$$\mathbf{g}' = \mathbf{g} - \mathbf{f}_0. \qquad \text{...(6)}$$

It is of some interest to note from the equation (4) that the force $\mathbf{F}_0$ acting at the constrained point is exactly the same when the point is moving, as when it is regarded as being at rest and the centrifugal force introduced. The reason is presumably that $\mathbf{F}_0$ is a physical quantity which must be the same to all observers.

4. *The equation of energy.*

Suppose that the external forces $\mathbf{F}$ are conservative, and are derivable from a potential function $\Omega(q_1, q_2, \dots q_n)$, where $q_1, q_2, \dots q_n$ are coordinates in the frame S_1 and are assumed not to contain the time t explicitly. We shall examine the conditions under which the energy of the relative motion is conserved.

We take the scalar product of the equation (1) with $\mathbf{v}_1$, and sum for all the elements m, thus obtaining

$$\Sigma(\mathbf{X} \cdot \mathbf{v}_1) = M(\mathbf{f}_0 \cdot \bar{\mathbf{v}}_1) + \frac{dT_1}{dt}, \qquad \text{..........................(7)}$$

where $T_1 = \frac{1}{2}\Sigma m\mathbf{v}_1{}^2$ is the kinetic energy of the relative motion, and $\Sigma(\mathbf{X} \cdot \mathbf{v}_1)$ gives the rate at which the external forces are doing work. Hence

$$\frac{d}{dt}(T_1 + \Omega) = -M(\mathbf{f}_0 \cdot \bar{\mathbf{v}}_1). \qquad \text{...........................(8)}$$

This equation integrates into

$$T_1 + \Omega = \text{constant}, \qquad \text{..................................(9)}$$

provided

$$(\mathbf{f}_0 \cdot \bar{\mathbf{v}}_1) = 0. \qquad \text{......................................(10)}$$

This happens in the following cases :

(i) if $\mathbf{f}_0 = 0$, that is, if the motion of O is one of uniform velocity ;

(ii) if $\bar{\mathbf{v}}_1 = 0$, that is, if either O and G coincide, or if the motion is a purely translatory motion ;

(iii) if the acceleration of O is directed towards or away from G.

That the energy of the relative motion is conserved in the case (i) can also be seen from the following considerations. The Newtonian equations of motion are valid for any frame of reference which is in uniform motion with respect to a fixed set of axes. Hence by considering the motion relative to a frame of reference moving with O, we see that all the equations of relative motion can be written down by supposing that O is at rest.

* Lamb, *Higher Mechanics*, Cambridge (1920), p. 154.

Our result for the case (ii) when O coincides with G shows that the motion relative to the centre of inertia is independent of the motion of the centre of inertia, which is d'Alembert's principle.

The third case is not of any practical interest.

5. Apart from the cases when the energy of the relative motion is conserved, that is, where the sum of the kinetic energy of the relative motion and the potential energy of the given external forces remains constant, there are also other interesting cases where there may exist certain energy integrals. These possibilities arise when $(\mathbf{f}_0 \cdot \bar{\mathbf{v}}_1)$ is a perfect differential. An important case in which this happens is when $\mathbf{f}_0$ is independent of the time t, that is, when the constrained point moves with uniform acceleration. Then the equation (8) is readily integrable, thus giving

$$T_1 + \Omega + M(\mathbf{f}_0 \cdot \bar{\mathbf{r}}_1) = \text{constant}. \quad \text{......(11)}$$

This may be written as

$$T_1 + \Omega_1 = \text{constant}, \quad \text{......(12)}$$

where

$$\Omega_1 = \Omega + M(\mathbf{f}_0 \cdot \bar{\mathbf{r}}_1) \quad \text{......(13)}$$

is the potential energy of the apparent forces.

We thus see that when the centrifugal force is a conservative field of force, then the sum of the kinetic energy of the relative motion and the apparent potential energy remains constant.

We illustrate the existence of this energy integral by a simple example.

Example 1. A point O of a plane lamina is moved in a horizontal straight line in the plane of the lamina with uniform acceleration f. Examine the energy integral.

If θ is the inclination of OG to the vertical, where G is the centre of inertia and $OG = h$, then the integral (12) gives

$$\tfrac{1}{2}I\dot{\theta}^2 - Mgh\cos\theta + Mfh\sin\theta = \text{constant}, \quad \text{......(14)}$$

where I is the moment of inertia about the point O. The sum of the kinetic energy of the relative motion and the potential energy of the external force does not remain constant, but the sum of the kinetic energy of the relative motion and the apparent potential energy is constant.

6. *The equations of motion in Lagrangian form.*

In some problems it will be easier to express the equations of motion in Lagrangian form than in the forms (4) and (5).

If T is the kinetic energy of the motion, then

$$T = \tfrac{1}{2}\Sigma m\mathbf{v}^2 = T_1 + M(\mathbf{v}_0 \cdot \bar{\mathbf{v}}_1) + \tfrac{1}{2}M\mathbf{v}_0^2. \quad \text{......(15)}$$

The equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = Q_r, \quad r = 1, 2, \dots n$$

may then be written in the form

$$\begin{aligned}\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_r} - \frac{\partial T_1}{\partial q_r} &= Q_r - M\frac{d}{dt}\frac{\partial}{\partial \dot{q}_r}(\mathbf{v}_0 \cdot \bar{\mathbf{v}}_1) + M\frac{\partial}{\partial q_r}(\mathbf{v}_0 \cdot \bar{\mathbf{v}}_1)\\ &= Q_r - M\left(\mathbf{f}_0 \cdot \frac{\partial \bar{\mathbf{v}}_1}{\partial \dot{q}_r}\right) - M\left\{\mathbf{v}_0 \cdot \left(\frac{d}{dt}\frac{\partial \bar{\mathbf{v}}_1}{\partial \dot{q}_r} - \frac{\partial \bar{\mathbf{v}}_1}{\partial q_r}\right)\right\}\\ &= Q_r - \frac{\partial}{\partial q_r}M(\mathbf{f}_0 \cdot \mathbf{r}_1). \quad \text{......(16)}\end{aligned}$$

Hence the Lagrangian equations of the relative motion may be written down by supposing that the point O is at rest, and adding the expression $M(\mathbf{f}_0 \cdot \bar{\mathbf{r}}_1)$ to the potential energy of the system.

From these Lagrangian equations of motion the energy equation can be simply obtained by multiplying each equation (16) by $\dot{q}_r$ and summing for all r from 1 to n. Then we have

$$\frac{d}{dt}(T_1+\Omega) = -M\dot{q}_r\frac{\partial}{\partial q_r}(\mathbf{f}_0 \cdot \bar{\mathbf{r}}_1) = -M(\mathbf{f}_0 \cdot \bar{\mathbf{v}}_1).$$

As an illustration of this method of solution we consider the following example.

Example 2. An end A of a rod AB of length $2a$ is made to describe a horizontal circle of radius b and centre O with uniform velocity $b\omega$. Investigate the motion of the rod and verify that a steady motion is possible with the plane OAB vertical and AB making an angle α with the vertical, where

$$\left(\frac{4}{3}a\sin\alpha + b\right)\omega^2 = g\tan\alpha.$$

If θ, ϕ denote the angular coordinates of the rod relative to A, then

$$T_1 = \frac{2}{3}Ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta),$$

$$\Omega = -Mga\cos\theta,$$

$$(\mathbf{f}_0 \cdot \bar{\mathbf{r}}_1) = -ab\omega^2\sin\theta\cos(\phi - \omega t).$$

The Lagrangian equations give

$$\frac{4}{3}a\ddot{\theta} - \frac{4}{3}a\sin\theta\cos\theta\,\dot{\phi}^2 = -g\sin\theta + b\omega^2\cos\theta\cos(\phi-\omega t),$$

$$\frac{4}{3}a\frac{d}{dt}(\sin^2\theta\dot{\phi}) = b\omega^2\sin\theta\sin(\phi-\omega t).$$

Steady motion is then seen to be possible if $\phi = \omega t$ and $\theta = \alpha$, provided

$$-\frac{4}{3}a\sin\alpha\cos\alpha\omega^2 = -g\sin\alpha + b\omega^2\cos\alpha,$$

that is, provided

$$\left(\frac{4}{3}a\sin\alpha + b\right)\omega^2 = g\tan\alpha.$$

Since $\phi = \omega t$, the plane OAB remains vertical. Hence the result.

University of Ceylon,
Colombo. C. J. E.

1492. Now the human body, in general, is neither much lighter nor much heavier than the water of the Seine; that is to say, the specific gravity of the human body, in its natural condition, is about equal to the bulk of fresh water which it displaces.—E. A. Poe, *The mystery of Marie Rogêt.* [Per Professor E. H. Neville.]

1493. Like an island city there stretched before my goggling eyes the mighty acreage of these docks—1000 acres to be precise, covering in all one and a quarter square miles.—Collie Knox, *The Unbeaten Track*, p. 128, on Cardiff docks. [Per Mr. G. L. Parsons.]

A PROPERTY CHARACTERISTIC OF QUADRICS OF REVOLUTION AND GENERAL CYLINDERS.

By E. D. Camier.

I. The locus of the centres of spherical curvature of a singly infinite family of geodesics which pass through a regular point O on a surface S, one in each direction in the tangent plane there, is, in general, a twisted curve. It will be proved that the only real surfaces, at all points of which (excluding umbilics) this locus is a plane curve, are quadrics of revolution and general cylinders.

II. Suppose S to be referred to its lines of curvature, and assume a rectangular system of axes $OXYZ$, with origin at O and the surface normal as Z axis, the X and Y axes being tangent to the coordinate curves through O.

For any curve $\mathbf{r}=\mathbf{r}(s)$ the position vector $\mathbf{C}$ of the centre of spherical curvature is given by

$$\mathbf{C}=\mathbf{r}+R\mathbf{n}+T\frac{dR}{ds}\mathbf{b},$$

where $\mathbf{n}$, $\mathbf{b}$ are the unit principal normal and binormal vectors, and R, T are the radii of curvature and torsion at the point in question, and where the arc s of the curve is taken as parameter.* Applying this to the geodesic $g(\theta)$ which makes an angle θ with the X axis at O, and supposing the axes to be suitably oriented, $R=1/k_n(\theta)$ and $T=1/t_g(\theta)$, where $k_n(\theta)$, $t_g(\theta)$ are the normal curvature and geodesic torsion in the direction θ. Moreover, taking position vectors with respect to O, $\mathbf{n}\equiv(0, 0, 1)$, $\mathbf{b}\equiv(\cos\theta, -\sin\theta, 0)$, and $\mathbf{r}\equiv(0, 0, 0)$. Then, referred to $OXYZ$, the centre of spherical curvature of $g(\theta)$ at O is given by

$$k_n{}^2t_gX=-\sin\theta\, dk_n(\theta)/ds_g,\quad k_n{}^2t_gY=\cos\theta\, dk_n(\theta)/ds_g,\quad Z=1/k_n, \quad \text{.......(1)}$$

s_g being the arc of the geodesic $g(\theta)$, supposed measured from O.

Now if k, $\bar{k}$ are the principal curvatures at O, and the arcs s, $\bar{s}$ are taken as parameters along the corresponding lines of curvature, it is known that

$$\left.\begin{aligned} &k_n(\theta)=k\cos^2\theta+\bar{k}\sin^2\theta,\quad t_g(\theta)=(\bar{k}-k)\cos\theta\sin\theta, \;\dots\dots\dots\dots\dots\\ &\text{and}\\ &dk_n(\theta)/ds_g=k_1\cos^3\theta+3k_2\cos^2\theta\sin\theta+3\bar{k}_1\cos\theta\sin^2\theta+\bar{k}_2\sin^3\theta, \;\dots\dots\dots \end{aligned}\right\}(2)$$

where $k_1=\partial k/\partial s$, $k_2=\partial k/\partial\bar{s}$, etc.†

Putting these expressions in (1) and letting θ vary, we have the equations of the locus considered.

If at O the surface is such that $dk_n/ds_g\equiv 0$ and k_n is not independent of θ, the locus degenerates into a portion of the normal there; but if at the same time k_n is constant ($t_g(\theta)\equiv 0$), *i.e.* O is an umbilic, the curve reduces to a point, though its coordinates are not given by (1). In the case $dk_n/ds_g\not\equiv 0$ and k_n constant (O also an umbilic) there is no locus at a finite distance, and this case will be excluded in what follows.

III. If the curve (1) is plane, a relation

$$AX+BY+CZ+D\equiv 0 \quad \text{..............................(3)}$$

must hold for all values of θ, where A, B, C, D are functions of the coordinates of O, such that $|A|+|B|+|C|=0$.

* Weatherburn, *Differential Geometry*, I, Art. 6.

† Darboux, *Théorie des Surfaces*, Art. 510. When the curve is a geodesic, $\bar{\omega}=\frac{\pi}{2}$, and the expression for "Laguerre's function" gives the equivalent of (2) since $\cos\theta=\sqrt{A}\,du/ds$, $\sin\theta=\sqrt{C}\,dv/ds$.

Substituting from (1) in (3) we get

$$(-A\sin\theta + B\cos\theta)\,dk_n(\theta)/ds + Ck_n(\theta)t_g(\theta) + Dk_n^2(\theta)t_g(\theta) \equiv 0. \quad \ldots\ldots\ldots (4)$$

Obviously if $t_g \equiv 0$ and $dk_n/ds_g \equiv 0$—the particular case mentioned above—(3) is satisfied. We assume $t_g \not\equiv 0$, *i.e.* $k \neq \bar{k}$.

The first two terms of (3), in virtue of (2), may be written together as a homogeneous polynomial in $\cos\theta$, $\sin\theta$, of the fourth degree, and as the relation is an identity, it follows that $D = 0$. Thus the plane must go through O.

Making the substitutions (2), expanding, and equating to zero the coefficients of the resulting polynomial of the fourth degree, the following equations are obtained :

$$\left.\begin{aligned} -\ Ak_1 + 3Bk_2 + (\bar{k} - k)kC &= 0, \\ -3A\bar{k}_1 + B\bar{k}_2 + (\bar{k} - k)\bar{k}C &= 0, \\ -A\bar{k}_2 = 0,\ Bk_1 &= 0, \\ -3Ak_2 + 3B\bar{k}_1 &= 0. \end{aligned}\right\} \quad (5)$$

(1) First suppose $C = 0$; then from (4) $dk_n/ds_g \equiv 0$, since $A = B = 0$ is now excluded. Any plane through the Z axis contains the curve, which is then a part of the normal at O.

(2) Next suppose that neither of k, $\bar{k}$ is zero. Then eliminating C between the first two of (5), we get with the remaining equations,

$$(3\bar{k}_1 k - k_1\bar{k})A + (3k_2\bar{k} - \bar{k}_2 k)B = 0,$$

$$-Ak_2 + B\bar{k}_1 = 0,$$

$$A\bar{k}_2 = 0, \quad Bk_1 = 0.$$

If (a) $k_1 = 0$, $\bar{k}_2 \neq 0$, *i.e.* $A = 0$, then $3k_2\bar{k} - k\bar{k}_2 = 0$ and $k_1 = 0$, and this means that, except for a constant, the factors of dk_n/ds_g are k_n and one factor of t_g. Essentially the same result is obtained on the assumption $\bar{k}_2 = 0$, $k_1 \neq 0$.

If (b) $k_1 = 0$, $\bar{k}_2 = 0$, then $\bar{k}_1 kA + k_2\bar{k}B = 0$ and $-k_2A + \bar{k}_1B = 0$, whence $\bar{k}_1^2 k + k_2^2\bar{k} = 0$. This is the condition that, apart from a constant, the factors of dk_n/ds_g should be t_g and one factor of k_n.

(3) In the same way, supposing one of k, $\bar{k}$, say k, to be zero, and examining all cases, we find that the only possible values for dk_n/ds_g are either

$$dk_n/ds_g \equiv 0, \text{ or } \equiv 3\bar{k}_1\cos\theta\sin^2\theta \text{ or } \equiv \bar{k}_2\sin^3\theta,$$

with similar forms when $\bar{k} = 0$.

Having found the conditions obtaining at the point O when the curve (1) is plane, we now consider what surfaces satisfy these at every point. We confine ourselves to real surfaces, and take the general cases 2 (a) and 2 (b) first.

The relations 2 (a), $k_1 = 0$, $\bar{k}_1 = 0$, $3k_2\bar{k} - k\bar{k}_2 = 0$, holding at every point of S are necessary and sufficient that it should be a quadric of revolution. For $3k_2\bar{k} - k\bar{k}_2 = 0$ is the same as $\partial\left(\frac{\bar{k}}{k^3}\right)/\partial\bar{s} = 0$, which with $\partial\left(\frac{k}{\bar{k}^3}\right)/\partial s = 0$ is the condition that S should be a quadric; * and the relations $k_1 = \bar{k}_1 = 0$ are characteristic of a surface of revolution.†

* Darboux, Art. 513.

† Blaschke, *Vorles. ü Differentialgeometrie*, I, p. 142.

In 2 (*b*), $k_1 = \bar{k}_2 = 0$ characterises the Cyclides of Dupin, in which the lines of curvature in both systems are circles; * but the relation $\bar{k}_1{}^2 k + k_2{}^2 \bar{k} = 0$ can only be true on special curves or at isolated points on such a surface.

Case 3 gives general cylinders, and includes (1) which is the circular cylinder.

Finally $dk_n/ds_g \equiv 0$, $t_g \equiv 0$ is characteristic of the sphere. In general at an umbilic $dk_n/ds_g \not\equiv 0$, and so there is no finite locus.

Summarising these cases, we have the result stated at the beginning of this note. It may also be shown without difficulty that the curve will be a cubic with a conjugate point at O.

The writer would be interested to know whether similar simple properties of this curve exist in the cases of other surfaces, *e.g.* developables, canal surfaces, etc. E. D. C.

THE VISUAL AIDS SUB-COMMITTEE.

THE first meeting of the Visual Aids Sub-Committee, elected by the Teaching Committee on 26th April, 1946, was held at 2.30 p.m. on 1st June, 1946, at Gordon House, Gordon Square, W.C. 1.

Messrs. Fairthorne, McCrea, Meredith, Pedoe, Woolley and Vesselo attended. Mr. I. R. Vesselo (80 Ventnor Drive, Totteridge, London, N. 20) was elected Chairman. Dr. D. Pedoe (36 Frederick Road, Edgbaston, Birmingham, 15) was elected Secretary.

There was an initial discussion on the scope of the term " Visual Aids ". Mr. Fairthorne submitted a memorandum, and it was agreed, as a temporary measure, that visual aids be listed in three categories:

1. Static. 2. Sequential. 3. Kinetic.

The next point was a discussion of the material at present available, and it was agreed that lists should be sent to Mr. G. P. Meredith, Visual Education Centre, University College, Exeter. These lists were to deal, if possible, with the question of cost and accessibility. Mr. Meredith was asked to pass these lists on to the Secretary, who would report, especially on the question of cost. The co-operation of members of the Mathematical Association was invited.

The Secretary was asked to enquire of the Control Commission for Germany as to the survival of factories which specialised in the manufacture of mathematical models.

Members of the Committee were then invited to submit memoranda on "The Place of Visual Aids in Teaching ".

There was a discussion on mathematical films, and it was suggested that contact should be established with the Services which used training films, and which might be interested in the production of mathematical films. It was agreed that research might well be undertaken on the production of mathematical films, and that such research would have to be subsidised full-time research. A number of suggestions for mathematical films already received were handed to the Secretary, for circulation amongst members of the Committee.

A possible Exhibition of Visual Aids at the next General Meeting of the Association was discussed, and Mr. Meredith agreed to undertake responsibility for this, and asked that suggestions should be sent to him.

The next meeting of the Committee was fixed for 2.30 p.m. on 21st September, 1946.

* Blaschke, p. 140.

MATHEMATICAL NOTES.

1900. ***Tétraèdres homothétiques.***

1. Soient deux tétraèdres homothétiques $T \equiv ABCD$ et $T' \equiv A'B'C'D'$; h_a, h_b, h_c, h_d les hauteurs du premier, d_a, d_b, d_c, d_d les distances des faces homologues, positives ou négatives suivant que le point A et la face $B'C'D'$, par exemple, sont du même côté du plan BCD ou non. Considérons les tétraèdres T_a', T_b', T_c', T_d' découpés par les plans des faces BCD, CDA, DAB, ABC de T dans les trièdres (A'), (B'), (C'), (D') de T'. Leurs rapports d'homothétie avec le tétraèdre T sont respectivement égaux à

$$\alpha = 1 - \frac{d_b}{h_b} - \frac{d_c}{h_c} - \frac{d_d}{h_d},$$

$$\beta = \dots, \quad \gamma = \dots,$$

$$\delta = 1 - \frac{d_a}{h_a} - \frac{d_b}{h_b} - \frac{d_c}{h_c}.$$

Il en résulte que

$$\alpha + \beta + \gamma + \delta = 4 - 3\Sigma(d_a/h_a) = 1 + 3\{1 - \Sigma(d_a/h_a)\} = 1 + 3k,$$

k étant le rapport d'homothétie de T' à T, car* $k = 1 - \Sigma(d_a/h_a)$.

Donc, *si le tétraèdre T' se déplace parallèlement à lui-même, la somme algébrique des longueurs de quatre segments rectilignes homologues dans T_a', T_b', T_c', T_d' reste constante et égale à $(1 + 3k)$ fois la longueur du segment rectiligne homologue des précédents, par rapport à T.* Par exemple, si r, r_a', r_b', r_c', r_d' sonts les rayons des sphères inscrites à T, T_a', T_b', T_c', T_d',

$$r_a' + r_b' + r_c' + r_d' = (1 + 3k)r = \text{const.}$$

2. Les droites AA', BB', CC', DD' concourent au centre d'homothétie H de T' et T et percent les faces BCD, CDA, DAB, ABC de T aux centres d'homothétie A_1, B_1, C_1, D_1 de T avec T_a', T_b', T_c', T_d'. Les droites joignant un point arbitraire associé à T à ses homologues dans T_a', ... passent donc par A_1, B_1, C_1, D_1. Dès lors, *la sphère (S_1) circonscrite au tétraèdre $A_1B_1C_1D_1$ touche les sphères (S_i'), $(i = a, b, c, d)$, qui lui sont homologues dans les tétraèdres T_i'*, aux points A_1, B_1, C_1, D_1.

En particulier, si le centre d'homothétie H coïncide avec le centre de gravité de T et T', on a cette proposition :

THÉORÈME. *Si deux tétraèdres sont homothétiques, par rapport à leur centre de gravité, la sphère des douze points † de l'un de ces tétraèdres touche les sphères des douze points des tétraèdres que les plans de ses faces détachent dans les quatre trièdres de l'autre.*

N.B. Ces remarques s'appliquent à deux triangles homothétiques et on retrouve ainsi une propriété énoncée par John Griffiths ‡ pour les cercles des neuf points d'un triangle ABC et des triangles BCA_1, CAB_1, ABC_1 formés par les parallèles à BC, CA, AB par A, B, C et les côtés du triangle fondamental.

3. Les conclusions sont les mêmes quand on prend T' pour triangle fondamental.

V. THÉBAULT.

* J. Neuberg, *The Tohoku Mathematical Journal*, 1914, p. 103.

† *Cfr.* N. A. Court, *Modern Pure Solid Geometry* (Macmillan, 1935), p. 250.

‡ *Nouvelles Annales de Mathématiques*, 1865, p. 322.

1901. ***Sur un triangle spécial en nombres entiers.***

Il s'agit du triangle ABC dans lequel les longueurs des côtés BC, CA, AB sont mesurées par les nombres entiers.

$$a=5m(2m+1), \quad b=4m(2m+1)+1, \quad c=(m+1)(6m+1),$$

m étant un entier positif arbitraire différent de zéro.

1. Comme on a simultanément

$$b^2+c^2>a^2, \quad c^2+a^2>b^2, \quad a^2+b^2>c^2,$$

les angles A, B, C sont tous aigus.

2. *Le côté moyen*

$$b=8m^2+4m+1=(2m)^2+(2m+1)^2$$

est la somme des carrés de deux nombres entiers consécutifs.

3. Avec les notations habituelles, on a successivement :

$$p=\tfrac{1}{2}(a+b+c)=(2m+1)(6m+1),$$

$$p-a=(m+1)(2m+1), \quad p-b=4m(m+1), \quad p-c=m(6m+1).$$

La surface

$$S=2m(m+1)(2m+1)(6m+1)$$

du triangle ABC est un nombre entier.

On a aussi

la hauteur $CC'=h_c=4m(2m+1)=b-1$; $2R=\frac{5}{4}b$;

$r=2m(m+1)=\frac{1}{2}(p-b)$; $r_a=2m(6m+1)=2(p-c)$; $r_b=\frac{1}{2}(2m+1)(6m+1)=\frac{1}{2}p$;

$r_c=2(m+1)(2m+1)=2(p-a)$; $r+r_b=\frac{1}{2}(c+a)$; $r_a+r_c=2b$; $r_cr_a=4rr_b=2S$.

Le pied C' de la hauteur $CC'=h_c$ partage le côté $AB=c$ en les segments rectilignes

$$AC'=4m+1 \quad \text{et} \quad C'B=3m(m+1).$$

N.B. Si $m=1$, on retrouve le triangle en nombres entiers *consécutifs* $a=15$, $b=13$, $c=14$, qui possède toutes les propriétés précitées et dans lequel on a en outre $h_c=r_c=12$. V. THÉBAULT.

1902. ***On the three-cusped hypocycloid.***

In his article (*Gazette*, 1945, p. 66) Prof. Hadamard recalls an earlier paper (1884), in which he proved that the asymptotes of any pencil of rectangular hyperbolas envelop a three-cusped hypocycloid, and he asks if that property was previously known.

The answer is affirmative. Although Steiner, in his fundamental article, "Ueber eine besondere Curve dritter Classe (und vierten Grades), *Journal für Mathematik*, vol. 53, 1856, pp. 231–7 (also *Gesammelte Werke*, 1882, vol. II, p. 639), did not actually state that the quartic, which is the envelope of the Simson lines of a triangle, is a three-cusped hypocycloid, he clearly mentioned (*Werke*, vol. II, p. 644) that the quartic may also be considered as the envelope of the asymptotes of a pencil of rectangular hyperbolas. The character of the Steiner curve as a three-cusped hypocycloid seems to have been first mentioned by Cremona, *Journal für Mathematik*, vol. 64, 1865, pp. 101–23, although the correspondence between Steiner and Schläfli, edited by J. H. Graf in 1896, seems to prove that the character of the curve was known to Schläfli or even to Steiner ; this is Prof. Loria's opinion (*Spezielle ebene Kurven*, vol. I, 1910, p. 160).

The Simson lines of the points where a circumdiameter of a triangle meets the circumcircle are perpendicular, and the well-known property about the segments determined on any straight line by an hyperbola and its asymptotes

proves that the considered Simson lines are the asymptotes of a rectangular hyperbola circumscribed to the triangle.

The just-recalled property about segments on a secant to an hyperbola is easy to prove geometrically (see, for instance, Prof. Hadamard's *Leçons de Géométrie*, vol. II, 1901, p. 448); the theorem on the envelope of the Simson lines has also often been proved geometrically (see, for example, H. Wieleitner, *Spezielle ebene Kurven*, 1908, p. 146). So the foregoing remarks give a geometric proof of the theorem considered by Prof. Hadamard, different from the one he develops in his paper. R. GOORMAGHTIGH.

1903. *The solution of the cubic equation.*

Take the equation

$$p + qx + rx^2 + x^3 = 0.$$

After having reduced the root by the approximation a, where $x = x_1 + a$, we write

$$p_1 + q_1x_1 + r_1x_1^2 + x_1^3 = 0, \qquad \text{(i)}$$

or, if $p_1 = q_1\alpha$, where α is known to be small,

$$-q_1\alpha = x_1(q_1 + r_1x_1 + x_1^2).$$

By reversion, writing $x_1 = -\alpha + k\alpha^2 + l\alpha^3 + \ldots$ and comparing coefficients,

$$q_1k + r_1 = 0,$$
$$q_1l - 2r_1k - 1 = 0.$$

Hence $$x_1 = -\frac{p_1}{q_1} - \frac{p_1^2r_1}{q_1^3} - \frac{2p_1^3r_1^2}{q_1^5} + \frac{p_1^3}{q_1^4} \ldots .$$

By equating coefficients with the continued fraction, as in my earlier Notes,

$$x_1 = \frac{p_1q_1}{p_1r_1 - q_1^2} - \left(\frac{p_1^3r_1^2 - p_1^3q_1}{q_1^5}\right).$$

Thus $$x = a + p_1q_1/(p_1r_1 - q_1^2), \qquad \text{(ii)}$$

with an error of about $(p_1^3r_1^2 - p_1^3q_1)/q_1^5$, or $x_1^3(r_1^2 - q_1)/q_1^2$ in excess.

Example. $x^3 - 2x - 5 = 0$.

Taking $a = 2{\cdot}1$, the equation corresponding to (i) is

$$\cdot061 + 11{\cdot}23x_1 + 6{\cdot}3x_1^2 + x_1^3 = 0.$$

Then $$p_1q_1/(p_1r_1 - q_1^2) = (\cdot061 \times 11{\cdot}23)/(\cdot061 \times 6{\cdot}3 - 11{\cdot}23^2)$$
$$= -\cdot00544848.$$

Hence $$x = x_1 + a = 2{\cdot}0945515,$$

correct to 7 decimal places, without taking into consideration the error term.

To arrive at this degree of accuracy by one application of Newton's iterative method, the initial value of a would have to be taken to be $2{\cdot}0946$. Also note that the number of decimals used is a minimum.

Note 1587, May 1942 (W. G. Bickley), can be much simplified in this way.

R. H. BIRCH.

1904. *Construction of tangent to an ellipse.*

The equation of a straight line cutting off the distances s and t on the axes of x and y respectively is

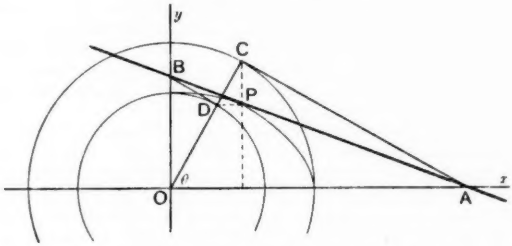
$$x/s + y/t = 1.$$

If the eccentric angle of a point on the ellipse is θ, the tangent is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1\,;$$

and, comparing these equations,

$$s=a/\cos\theta, \qquad t=b/\sin\theta.$$



In the right-angled triangle OAC, $\angle COA=\theta$, and $OC=a$, therefore

$$OA=a/\cos\theta=s,$$

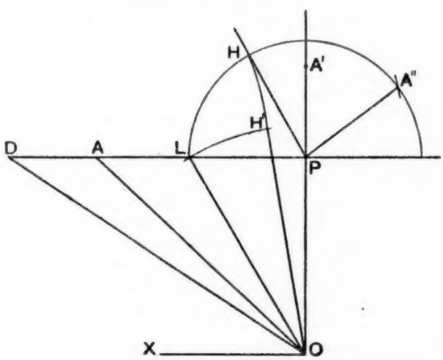
and from the right-angled triangle OBD, $OB=b/\sin\theta=t$. Thus the straight line joining AB touches the ellipse at P, the point whose eccentric angle is θ.

S. W.

1905. *A construction for the altitude and bearing of a star.*

In the figure,

$$\angle ODP=\text{declination of star},$$
$$\angle OLP=\text{latitude of observer},$$
$$\angle DPH=\text{hour angle of star}.$$



Now mark H such that $PH=PL$, and draw arcs, centre D and radius DH, centre O and radius OL to give H'. Then

the altitude of the star is $90°-\angle DOH'$.

Draw $\angle AOP = \angle DOH'$; mark A' on OP such that $OA' = OA$, then on the circle centre P find A'' such that $AA'' = DA'$. Then

$\angle APA''$ is the bearing of the star.

THOMAS W. GEE.

1906. ***Fictitious forces: their use and misuse.***

This protest against the use of wrong ideas, in fact the obscuring of true ideas, was originally made just after Mr. E. G. Phillips had submitted his protest. In it I gave what seemed to me the proper way to present the working, and this was exactly in agreement with his ideas. However, there are one or two points which may be added or stressed, and the following may be regarded as a " variation on an original theme ".

It is necessary to stress the difference between a force and its measurement. The numerical value of a force is not the force itself. Thus, in the oft-used equation $F = ma$, it is wrong to say that " ma " is a force. We may say that the expression has the same dimensions as a force, but it is obviously the product of a mass and an acceleration, and to say that a force of " ma " is acting is absurd. It is rather like saying that the result of a certain action is the cause of itself.

There are many who will agree with this and yet will teach ideas about " centrifugal force " without realising that they are giving false notions. It is not generally realised that these wrong ideas are taught explicitly or implicitly not only in schools but also in much higher places. I have been protesting in a small way about it for a number of years, and yet I find " centrifugal force " still shown in diagrams in textbooks.

The presentation to which I object is put something like this: If a body of mass m moves in a circle of radius r with uniform speed v, then it has an acceleration of v^2/r towards the centre. The product of mass and acceleration is mv^2/r, hence " there is a force of mv^2/r towards the centre ", *i.e.* " a centripetal force ". But, they say, by Newton's third law, there must therefore be a force of mv^2/r outwards, in other words " a centrifugal force ".

I would say that this is an utterly false presentation. The pupil will inform us that a stone whirling on a piece of string flies off when the string is cut " because of centrifugal force ". I would urge that any use of the word force should be reserved for such things as tension, friction, push of rails, etc., which do make the body in question move in a curve. Again, and perhaps most important of all, in the above work the fact that the body tends to move in the direction of the tangent to the curve (Newton's first law) is entirely obscured, in fact it is contradicted.

There is, then, no need to use the words " centrifugal and centripetal ". As Mr. Phillips has urged, their use is an abomination. It may be necessary for some time to come to explain how problems of this nature have been reduced to Statics problems, because our pupils are bound to see some reference to them. At the same time the teacher must say over and over again that it is not a Statics problem. All these problems can be dealt with quite simply by the use of dynamical principles. Added to this, the pupils will gain some idea of what is really happening, instead of using a shibboleth without thinking.

There may be some readers who are fortunate enough to live in an atmosphere where these wrong ideas do not occur and where the books containing them do not appear, but I can assure them that the wrong thinking does exist, as Mr. Phillips has shown. I would beg such people to use their influence, not least on the teachers, to cause the expression of such ideas to disappear.

T. A. GROCOCK.

1907. *Some unusual series occurring in n-dimensional geometry.*

Let us consider the number of regions into which a limitless plane is divided by m general straight lines, or a limitless space by m general planes, or in general a limitless n-dimensional manifold by m $(n-1)$-dimensional manifolds.

Our condition of generality is that in the case $n=2$, for instance, no three of the dividing lines should be concurrent and no two parallel. Similar conditions apply in the general case.

Designating the number of regions by $S_n(m)$, the most simple case is that in which $n=1$, giving the number of regions into which a line is divided by m points upon it.

Initially the line constitutes one whole region and each point inserted on it produces one new region.

Obviously

$$S_1(m)=1+m. \quad \text{.....................................(I)}$$

To determine $S_2(m)$ we may note that the $(m+1)^{st}$ line cuts each of the previous m lines once and is thus divided into $S_1(m)$ regions.

Each of the regions of this line cuts a region of plane into two, and thus altogether the $(m+1)^{st}$ line produces $S_1(m)=(1+m)$ new regions of plane.

Since, furthermore, the original plane has one region we have

$$S_2(m)=1+\sum_0^{m-1} S_1(m) \quad \text{................................(II)}$$

$$=1+\sum_0^{m-1}(1+m)$$

$$=1+\tfrac{1}{2}m(m+1). \quad \text{...........................(III)}$$

The same type of argument can be used in the general case, giving an equation that can be used inductively :

$$S_{(n+1)}(m)=1+\sum_0^{m-1} S_n(m). \quad \text{...........................(IV)}$$

This function has been tabulated for a range of values of n and m.

It can easily be seen that any term in the table is the sum of two preceding ones :

$$S_n(m)=S_n(m-1)+S_{n-1}(m-1). \quad \text{.......................(V)}$$

This together with the condition

$$S_n(0)=S_0(m)=1 \quad \text{................................(VI)}$$

is sufficient to define the function.

It will be noted that $S_n(m)=2^m$ so long as $m \leqslant n$. For values of m larger than this the simple relationship breaks down.

For instance, at $m=n+1$ we have $S_n(n+1)=2^{n+1}-1$.

The geometrical reason for this breakdown of the simple law is not hard to find.

In the two-dimensional case the number of regions is doubled with each added line until three lines have been added, in which case a closed figure, a triangle, is formed. Similarly for $n=3$, the fourth plane leads to the formation of a tetrahedron, and this closed figure may be looked on as being equivalent to two regions given by the expression 2^4. These are coincident, and hence the real number of regions is $S_3(4)=2^4-1$.

From equations (III) and (IV) it will be seen that $S_n(m)$ involves the summation of the $(n-1)^{st}$ powers of the first m numbers, and hence $S_n(m)$ is a polynomial of the nth degree in m. This polynomial may be defined by its

first $n+1$ values which, as have been seen, obey the law $S_n(m)=2^m$. This is sufficient to calculate the coefficients of the polynomial.

E.g., in the case $n=3$, if we put

$$S_3(m)=Am^3+Bm^2+Cm+D, \quad \text{(VII)}$$

and remember that

$$S_3(0)=1,$$
$$S_3(1)=2,$$
$$S_3(2)=4,$$
$$S_3(3)=8,$$

we have by substitution and elimination :

$$A=\tfrac{1}{6},\ B=0,\ C=\tfrac{5}{6},\ D=1.$$

It is interesting to note from the table that

$$S_n(2n+1)=2^{2n}. \quad \text{(VIII)}$$

Since the $(n+1)^{\text{st}}$ difference of $S_n(m)$ is zero and the "end differences" are all unity, we have by the principle of finite differences :

$$S_n(m)=1+\frac{m}{\underline{|1}}+\frac{m(m-1)}{\underline{|2}}+\ldots\ldots+\frac{m(m-1)\ldots(1+m-n)}{\underline{|n}}. \quad \text{(IX)}$$

If $m \leqslant n$ this expression becomes simply the binomial expansion :

$$(1+1)^m=2^m,$$

so proving the previously noted law.

For the case $m=2n+1$ noted above, it will be seen from (XI) that just the first half of the binomial expansion occurs, giving

$$S_n(2n+1)=\tfrac{1}{2}2^{2n+1}=2^{2n},$$

thus proving this special case.

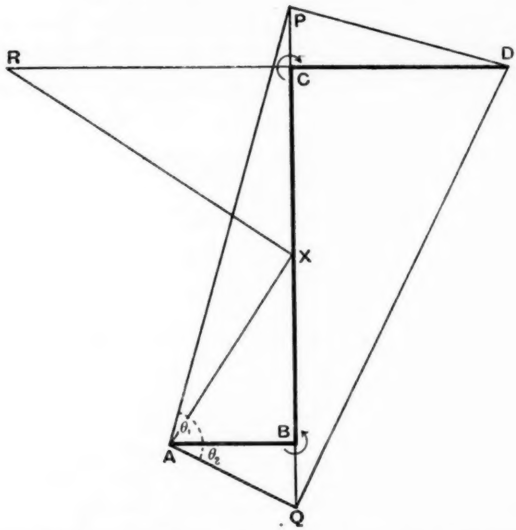
These series are useful to illustrate the point that a series derived from quite simple principles may follow a different law from that suggested by an inspection of the first few terms. In this case the number of terms following the 2^m law (*i.e.* in geometrical progression) may be made as large as is desired by taking an appropriate value of n.

m	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
0	1	1	1	1	1	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2	2	2
2	1	3	4	4	4	4	4	4	4	4	4
3	1	4	7	8	8	8	8	8	8	8	8
4	1	5	11	15	16	16	16	16	16	16	16
5	1	6	16	26	31	32	32	32	32	32	32
6	1	7	22	42	57	63	64	64	64	64	64
7	1	8	29	64	99	120	127	128	128	128	128
8	1	9	37	93	163	219	247	255	256	256	256
9	1	10	46	130	256	382	466	502	511	512	512
10	1	11	56	176	386	638	848	968	1013	1023	1024
11	1	12	67	232	562	1024	1486	1816	1981	2036	2047
12	1	13	79	299	794	1586	2510	3302	3797	4017	4083
13	1	14	92	378	1093	2380	4096	5812	7099	7814	8100
14	1	15	106	470	1471	3473	6476	9908	12911	14913	15914

Derek J. Price.

1908. *Solution of quadratic equations.*

A method devised by Lill for finding graphically the real roots of an equation has the merit that, apart from the absence of arithmetic, when applied to a quadratic equation a ruler and set-square solution is obtainable.



For example, let the equation be $2x^2 - 6x - 3{\cdot}5 = 0$. The coefficients 2, 6, 3·5 are plotted successively at right angles, clockwise to each other, save when a change of sign occurs: $AB = 2$ units, $BC = 6$ units, $CD = 3\frac{1}{2}$ units. Then P, Q are found on BC such that $\angle APD$, $\angle AQD$ are right angles. This can be done by set-square, or by drawing the circle on AD as diameter. The latter method will give greater accuracy.

Now $$BP = 6 + CP = 2 \tan \theta_1,$$

$$CD = 3{\cdot}5 = PC \tan \theta_1 = (2 \tan \theta_1 - 6) \tan \theta_1,$$

and hence $$3{\cdot}5 = 2 \tan^2 \theta_1 - 6 \tan \theta_1.$$

Thus the roots are $\tan \theta_1$, $\tan \theta_2$; in this case 3·5, $-0{\cdot}5$.

The minimum value of the expression is given when θ is the angle which XA makes with AB, where X is the midpoint of BC; in this case, $\tan \theta = 1{\cdot}5$. The value of the expression is then given by RD, where XR is at right angles to AX.

H. G. MIDDLETON.

1909. *Notes on Conics.* **12:** *What was Pascal's own proof of his theorem?*

Writing of Pascal's theorem in his *History of the Conic Sections and Quadric Surfaces*, Professor Coolidge says (p. 33): "How did Pascal prove it? We have no faintest clue. He points out that as it is a projective theorem, if true of a circle, it is also true of a general conic. This suggests that he first proved it for the circle, though I know of no proof for this special case that is as easy as the best proofs for the general case."

That Pascal proved the theorem first for the circle is not a matter of dubious inference. Lemma I of the famous Essay asserts the theorem for the circle, Lemma II asserts, in effect, that any theorem on concurrence of lines is a projective theorem, and then the theorem is said to be demonstrable for the conic on the basis of these two lemmas, and to constitute in this general form a third lemma.

Prof. Coolidge has forgotten one proof for the circle which is definitely more elementary than any of the general proofs, namely, the proof by quadruple use of Menelaus' theorem. If, in Pascal's own figure, the line NO cuts AM, AS, PQ in J, U, W, we have, from collinearities on the sides of the triangle UJA,

$$\frac{JM}{MA} \cdot \frac{AV}{VU} \cdot \frac{UO}{OJ} = -1,$$

$$\frac{JK}{KA} \cdot \frac{AS}{SU} \cdot \frac{UN}{NJ} = -1,$$

$$\frac{JP}{PA} \cdot \frac{AQ}{QU} \cdot \frac{UW}{WJ} = -1;$$

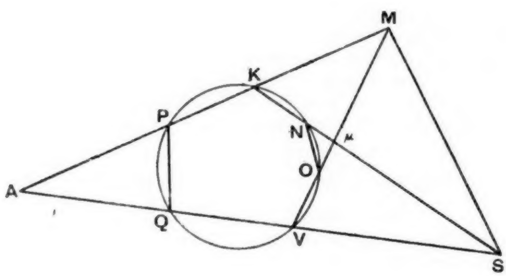
multiplying, and using the equalities

$$JK \,.\, JP = OJ \,.\, NJ, \quad AV \,.\, AQ = KA \,.\, PA, \quad UO \,.\, UN = VU \,.\, QU,$$

we have

$$\frac{JM}{MA} \cdot \frac{AS}{SU} \cdot \frac{UW}{WJ} = -1,$$

and this is the condition of collinearity of M, S, W, that is, of concurrence of MS, NO, PQ. The corresponding proof for the general conic requires Carnot's theorem, instead of merely the rectangle property of the circle.



This elementary proof, in the form in which I have just given it, can not be the one that Pascal found, for it is unlikely that the triangle I have called AJU would have disappeared from the figure if it had played in fact the fundamental role, and it is certain that the result could not have been presented unsymmetrically if the demonstration had had this perfect symmetry. But, by the same arguments, it is plausible that Pascal used Menelaus' theorem to determine the intersection of NO with MS from the triangle μMS and the intersection of PQ with MS from the triangle AMS, for in the text of the essay no use is made of the point marked μ.

A possible verification runs as follows. First, if NO cuts MS in W, then

$$\frac{MW}{SW}=\frac{\mu N}{\mu O}\cdot\frac{MO}{SN}$$

$$=\frac{\mu V}{\mu K}\cdot\frac{MK\,.\,MP\,.\,SK}{MV\,.\,SV\,.\,SQ},$$

the rectangle property being invoked for the elimination of the points N and O. But

$$\frac{\mu V\,.\,SK}{\mu K\,.\,MV}=\frac{\Delta SKV}{\Delta MKV}=\frac{VS\,.\,AK}{KM\,.\,AV}.$$

Hence

$$\frac{MW}{SW}=\frac{AK\,.\,PM}{AV\,.\,QS},$$

and eliminating K and V by another use of the rectangle theorem, we have

$$\frac{MW}{SW}=\frac{AQ\,.\,PM}{AP\,.\,QS},$$

establishing that W is on PQ.

This proof has the merit of reaching precisely the lemma which Pascal enunciated, by means of theorems with which Pascal was certainly familiar, in a figure which Pascal actually drew. It is, as I have called it, a verification. But what more can we ask for? The genius of the discovery lay in suspecting that the theorem might be true, and who will be so bold as to account ingloriously for the glorious conjecture?

Pascal's own figures, drawn for an ellipse, are reproduced in the *Gazette*, XII, 53 (1924), where J. J. Milne gave a translation of the *Essay*. Another translation, by F. M. Clarke, was published in *Isis*, X, 33, and in revised form in D. E. Smith's *Source Book in Mathematics*, 326 (1929). The figure in this Note retains the original lettering, but includes only the lines and points relevant to Lemma I.

The deduction of Pascal's theorem from Carnot's was given in Milne and Davis' *Geometrical Conics*, 198 (1894), and may well be much older, for it is very obvious.

E. H. N.

1910. *On* 4×4 *pan-magic squares.*

From the pattern of complements given in Nancy and W. J. Chater's article in the *Mathematical Gazette*, July, 1945, the total number of essentially different 4×4 pan-magic squares can be deduced and is found to be surprisingly small. The pattern of complements for such squares, as given on page 96, but with dashes added to distinguish the two members of the pair of complements is :

a	h	e	c
g	b	d	f
e'	c'	a'	h'
d'	f'	g'	b'

where $a+a'=b+b'=\ldots=17$.

Since the square may be regarded as a cut from a magic parquet, there is no loss in generality in assuming $a=1$. Hence $a'=16$.

Since $a+b+g+h=34$, and b, g and h are each <16, they are each >3. Similarly, reading up the parquet, $a+b'+c+d'=34$, hence b', c and d' are each >3. Now, if $b=14$ or 15, then $b'=3$ or 2, which is excluded. Hence

$$3<b<14.$$

Now any square with $b>8$ can be changed into one with $b<8$ by permuting rows and columns cyclicly, so that the first row and first column become the last row and last column respectively, so that b' takes the place of b. Hence, without loss of generality, we need only consider $b=4$, 5, 6, 7, and 8.

Also without loss of generality we can take $h>g$. The only possible sets of values for (a, b, g, h) are therefore

(1, 4, 14, 15)	(1, 6, 13, 14)	(1, 8, 10, 15)
(1, 5, 13, 15)	(1, 7, 11, 15)	(1, 8, 11, 14)
(1, 6, 12, 15)	(1, 7, 12, 14)	(1, 8, 12, 13)

For each of these sets, we can find possible sets of values for c and d' since $a+b'+c+d'=34$. For example, for (1, 4, 14, 15) we have (8, 12); (12, 8); (9, 11), and (11, 9). In all, it is found that there are 24 sets of possible values of (a, b, g, h, b', c, d').

Consider the 3×3 portion of the parquet

b'	d'	f'
c	a	h
f	g	b

We can attempt to find values for f and f' for each of the 24 sets of possible values of (a, b, g, h, b', c, d'). Of these 24 cases:

10 cases do not provide possible values for both f and f'.

7 cases duplicate ones already found, but rotated about a.

4 cases contain rows or columns such as (1, 10, 13) or (1, 9, 15) for which no fourth number can be added to make 34.

There are thus only three satisfactory solutions, and hence only three essentially different 4×4 pan-magic squares, namely:

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

A

1	15	6	12
14	4	9	7
11	5	16	2
8	10	3	13

B

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

C

The example given on page 96 of Nancy and W. J. Chater's article is *B* with rows and columns interchanged and permuted cyclicly.

D. M. Hallowes.

1911. *Pan-magic squares.*

Nancy and W. J. Chater in their analysis of magic squares * showed that a pan-magic square of order 4 may be represented by

* *Math. Gazette*, XXIX, No. 285 (July, 1945).

a	h	e	c
g	b	d	f
e'	c'	a'	h'
d'	f'	g'	b'

where a, a' ; b, b' ; ... , are complementary ; *i.e.* if S is the " square constant ",

$$(a+a')=(b+b')=\ldots=\tfrac{1}{2}S,$$

and the sum of each row, column and diagonal is S.

Writing $(\frac{1}{2}S-a)$ for a', $(\frac{1}{2}S-b)$ for b', etc., we may form the determinant

$$\Delta=\begin{vmatrix} a & h & e & c \\ g & b & d & f \\ \frac{1}{2}S-e & \frac{1}{2}S-c & \frac{1}{2}S-a & \frac{1}{2}S-h \\ \frac{1}{2}S-d & \frac{1}{2}S-f & \frac{1}{2}S-g & \frac{1}{2}S-b \end{vmatrix}$$

Forming a new top row by adding the second and subtracting the third and fourth rows, we have

$$\Delta=\begin{vmatrix} (a+g+e+d)-S & (h+b+c+f)-S & (e+d+a+g)-S & (c+f+h+b)-S \\ g & b & d & f \\ \frac{1}{2}S-e & \frac{1}{2}S-c & \frac{1}{2}S-a & \frac{1}{2}S-h \\ \frac{1}{2}S-d & \frac{1}{2}S-f & \frac{1}{2}S-g & \frac{1}{2}S-b \end{vmatrix}$$

and since $(a+g+e+d)=(h+b+c+f)=\ldots=S$,

$$\Delta=0.$$

This property appears to be peculiar to pan-magic squares of order 4.

A. H. MacColl.

1912. *Integration by substitution.*

The familiar formula for integration by substitution,

$$\int_a^b f(x)\,dx=\int_{t_0}^{t_1} f\{g(t)\}\,g'(t)\,dt,$$

is often mistakenly supposed to be invalid if the derivative $g'(t)$ changes sign (or just vanishes) in the range of integration (t_0, t_1) ; this supposition in turn is based upon the false assumption that the substitution relation $x=g(t)$ must establish a (1, 1) correspondence.

Sufficient conditions, independent of the sign of the derivative $g'(t)$, are given by de la Vallée Poussin, *Cours d'Analyse Infinitesimale*, I, p. 214, as follows :

(1) $f(x)$ is continuous in (a, b) ;

(2) $g(t_0)=a,\ g(t_1)=b$;

(3) $g(t)$ has a continuous derivative in (t_0, t_1) ;

(4) $f\{g(t)\}$ is continuous in (t_0, t_1).

Condition (4) is needed only if $g(t)$ takes values outside (a, b) when t varies from t_0 to t_1.

The object of the present note is to draw attention to a deficiency in de la Vallée Poussin's proof of the substitution formula under these conditions

and to establish the theorem needed to complete this proof. The proof given in the *Cours d'Analyse* proceeds by observing that if t lies in (t_0, t_1) then the two functions of t,

$$\int_a^{g(t)} f(x)\,dx \quad \text{and} \quad \int_{t_0}^{t} f\{g(t)\}\,g'(t)\,dt,$$

have the same derivative $f\{g(t)\}g'(t)$. For the second integral, conditions (3) and (4) are obviously sufficient to ensure that the integrand is the derivative of the integral; in the case of the first integral we require to know that $f(x)$ is continuous in the range $(a, g(t))$ for *any* t in (t_0, t_1). At first sight one might suppose that this is merely another way of stating condition (4), but more careful consideration shows that the connection is far from immediate.

We have to prove that if $g(t)$ and $f\{g(t)\}$ are continuous in (t_0, t_1) and if α and β are the lower and upper bounds of $g(t)$ in this interval, then $f(x)$ is continuous in (α, β).

Suppose, on the contrary, that $f(x)$ is discontinuous at a point x_0 in (α, β); then we can determine a positive integer k and a sequence of points $x_1, x_2, x_3, \ldots$, all lying in (α, β) and converging to x_0, such that

$$|f(x_n) - f(x_0)| > 1/k, \quad n \geqslant 1.$$

Since $g(t)$ is continuous in (t_0, t_1) and x_n lies between the values α and β of the function $g(t)$, we can determine τ_n in (t_0, t_1) such that $g(\tau_n) = x_n$. The bounded set τ_n, $n \geqslant 1$, has at least one limiting point τ, say, and we can select a subset $\tau_{n(r)}$, say, such that $\tau_{n(r)}$, $r = 1, 2, 3, \ldots$, converges to τ. Then, as $g(t)$ is continuous, $g(\tau_{n(r)})$ converges to $g(\tau)$, *i.e.* $x_{n(r)}$ converges to $g(\tau)$; but $x_{n(r)}$ converges to x_0 and therefore $g(\tau) = x_0$. Hence

$$|f\{g(\tau_{n(r)})\} - f\{g(\tau)\}| = |f(x_{n(r)}) - f(x_0)| > 1/k,$$

which proves that $f\{g(t)\}$ is discontinuous at τ. This contradiction establishes the theorem.

The need for this result may be obviated by stating the conditions under which the substitution formula holds in the following form:

(1) $g(t_0) = a$, $g(t_1) = b$; (2) in the interval (t_0, t_1), $g(t)$ is continuous and $\alpha \leqslant g(t) \leqslant \beta$; (3) $f(x)$ is continuous in (α, β).

The continuity of $f(x)$ in $(a, g(t))$ is ensured by condition (3), and the continuity of $f\{g(t)\}$ in (t_0, t_1) is a trivial deduction from conditions (2) and (3). R. L. GOODSTEIN.

1913. *Linkage puzzles.*

A familiar topological puzzle asks for the number of paths between two marked points connected by a maze of intersecting lines. More precisely, a finite number of points are joined, each to each, by line segments, and the problem is to find the number of simple contours, formed of line segments, connecting one chosen point with another. We shall confine ourselves here to the simplest case in which each line segment may be supposed to carry an arrow indicating the unique direction in which this segment may be traversed; in practice this restriction is effected by prohibiting any move which has a positive component *towards* the starting point.

An obvious solution is obtained by assigning, *progressively*, to each point of the configuration the number of the totality of paths leading to it from the starting point, in the following manner:

The starting point is numbered 1; any other point bears the number which is the sum of the numbers at the far ends of all the line segments *leading* to this point.

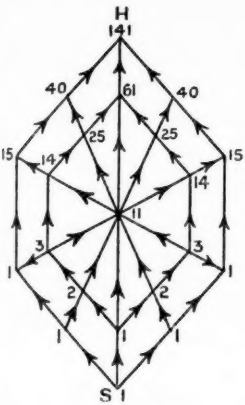


FIG. 1.

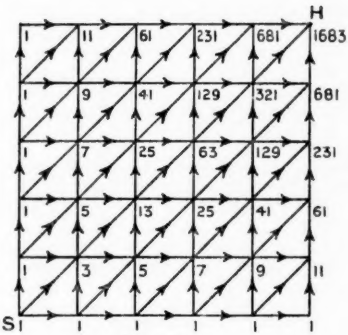


FIG. 2.

The method is illustrated in the two diagrams ; in each S denotes the start, and H is home.

R. L. G.

1914. *Note on a continued fraction.*

After the correspondence in a recent issue of the *Gazette*, I feel a little dubious about putting forward the following method of expressing a fraction as a continued fraction and *vice versa*. However, there may be some who will find the method new and useful.

It is better to explain by examples.

Example 1. Express $\frac{113}{89}$ as a continued fraction.

Write 89 below 113 and subtract as many multiples of 89 from 113 as possible, writing the number of multiples of 89 on the left and the result of subtraction under the 89 thus :

$$\begin{array}{r|r} & 113 \\ 1 & 89 \\ & 24 \end{array}$$

Repeat with 89 and 24 in place of 113 and 89, and continue until the result of subtraction is 1. We then have the following array :

$$\begin{array}{r|r} & 113 \\ 1 & 89 \\ 3 & 24 \\ 1 & 17 \\ 2 & 7 \\ 2 & 3 \\ & 1 \end{array}$$

The list of multiples starting from the top, together with the figure above the final 1, are the figures to be used in writing down the continued fraction, thus :

$$\frac{113}{89}=1+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3}.$$

Example 2. 3 $$f = 2 + \frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{2}.$$

The working is shown in brackets on the right.

$$\begin{array}{r|rl} & 81 & (=2\,.\,35+11) \\ 2 & 35 & (=3\,.\,11+\ 2) \\ 3 & 11 & (=5\,.\ \ 2+\ 1) \\ 5 & 2 & \\ & 1 & \end{array}$$

Then $$f = \frac{81}{35}.$$

Example 3. 4 $$f = \frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}.$$

$$\begin{array}{r|r} & 57 \\ 5 & 10 \\ 1 & 7 \\ 2 & 3 \\ & 1 \end{array} \qquad f = \frac{1}{57/10} = \frac{10}{57}.$$

Arising from this method of writing the working, there is a quick way of arriving at the solution in integers of equations of the type $ax + b = cy + d$, a, b, c, d being given integers.

Example ~~4~~ 2. $\frac{59}{11}$. The array is

$$\begin{array}{r|r} & 59 \\ 5 & 11 \\ 2 & 4 \\ 1 & 3 \\ & 1 \end{array}$$

and $$\frac{59}{11} = 5 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3}.$$

When the numerator of the given fraction f is less than the denominator, we still start with the larger number and in fact express $1/f$ as the continued fraction.

Thus $$\frac{89}{113} = \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3}.$$

The reverse process is easily carried out.

Given $$f = b_1 + \frac{1}{b_2+}\,\frac{1}{b_3+}\cdots\frac{1}{+b_{n-2}+}\,\frac{1}{b_{n-1}+}\,\frac{1}{b_n},$$

write down the array as follows on the left and work up as shown on the right:

$$\begin{array}{l|l} b_1 & \\ b_2 & \\ . & \\ . & \\ b_{n-2} & \\ b_{n-1} & b_n \\ & 1 \end{array} \qquad\qquad \begin{array}{l|l} b_1 & . \\ . & . \\ . & . \\ . & b_{n-2}(b_{n-1}b_n+1)+b_n \\ b_{n-2} & b_{n-1}b_n+1 \\ b_{n-1} & b_n \\ & 1 \end{array}$$

It is well known that such equations can be solved by using the theorem $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ where, in the usual notation, p_n/q_n is the nth convergent of a continued fraction (see, for example, Hall and Knight's *Higher Alegbra*). Effectively this involves expressing the penultimate convergent of a continued fraction as a vulgar fraction. Let us take an example.

Example 5. Find k and h so that

$$362k = 131h + 1.$$

Write down the array for expressing $\dfrac{362}{131}$ as a continued fraction.

$$\begin{array}{r|r} & 362 \\ 2 & 131 \\ 1 & 100 \\ 3 & 31 \\ 4 & 7 \\ 2 & 3 \\ & 1 \end{array}$$

The penultimate convergent is obtained by omitting the last of the figures 2, 1, 3, 4, 2, 3, and finding the vulgar fraction corresponding to

$$2 + \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{2}.$$

The working for this is as in Example 3, and the values of h and k are the figures at the top of the column, so that

$$38 \,.\, 362 = 105 \,.\, 131 + 1.$$

$$\begin{array}{r|r} & 105 \\ 2 & 38 \\ 1 & 29 \\ 3 & 9 \\ 4 & 2 \\ & 1 \end{array}$$

We were able to obtain $+1$ because the number of multiples written down in expressing 362/131 as a continued fraction was odd. If the number of multiples is even, the $+1$ becomes -1. This can always be arranged. For instance, in Example 5, if instead of stopping at the stage $2 \left| \begin{array}{l} 3 \\ 1 \end{array} \right.$, we had made one more step by taking two 1's from the 3, our array would be

$$\begin{array}{r|r} & 362 \\ 2 & 131 \\ 1 & 100 \\ 3 & 31 \\ 4 & 7 \\ 2 & 3 \\ 2 & 1 \\ & 1 \end{array}$$

Then the second part of the solution would start with the framework :

$$\begin{array}{r|r} & \\ 2 & \\ 1 & \\ 3 & \\ 4 & \\ 2 & 2 \\ & 1 \end{array} \qquad \text{This leads to} \qquad \begin{array}{r|r} & 257 \\ 2 & 93 \\ 1 & 71 \\ 3 & 22 \\ 4 & 5 \\ 2 & 2 \\ & 1 \end{array}$$

and we have $93 \,.\, 362 = 257 \,.\, 131 - 1$.

The solution for a case like $ax = cy + d$ $(1 < d < c)$ can easily be obtained.

Example 6. Find k if $31k = 17 \pmod{23}$.

The working is shown below.

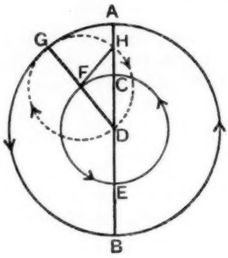
$$\begin{array}{r|r} & 31 \\ 1 & 23 \\ 2 & 8 \\ 1 & 7 \\ & 1 \end{array} \qquad \begin{array}{r|l} & 4 \\ 1 & 3 \times 17 = 51 = 5 \pmod{23}. \quad \text{Hence} \quad k = 5. \\ 2 & 1 \\ & 1 \end{array}$$

F. Ayres.

1915. *Simple harmonic motion in the sixteenth century.*

An oscillation of the type now termed "simple harmonic" was precisely

defined by the sixteenth-century Polish astronomer Copernicus in his historic book *De Revolutionibus* (1543, Bk. III, §§ 3–8). Copernicus believed that each equinoctial point (intersection of celestial equator and ecliptic) oscillated about its mean position with a motion which he first characterises in general terms as occurring " after the manner of suspended bodies . . . over the same course, between two limits, most rapidly in the middle . . . and slowest at the ends ". He then defines the motion more exactly as follows : With centre D, draw two concentric circles, AB, CE, the outer circle having twice the radius of the inner. With centre F (any point on the inner circle) and radius FD,



draw a circle cutting the diameter AB in H and D. Make F describe the circle CFE at a uniform rate ; and make H describe the circle GHD at twice the rate of F and in the opposite sense of rotation. It can then be shown that H moves backwards and forwards along the line AB, and that, in modern notation, $DH = DG \,.\, \cos GDH$, so that the motion of H on AB is simple harmonic. Copernicus supposed the equinoctial point to oscillate in this manner to and fro on the ecliptic about its mean position D, with a period, amplitude and epoch determined from observations ; and he worked out its position on any given date just as we calculate the displacement of the moving point from a knowledge of the amplitude and phase of the simple harmonic motion.

A. Armitage.

1916. *Archimedes's Principle and Newton's Third Law.*

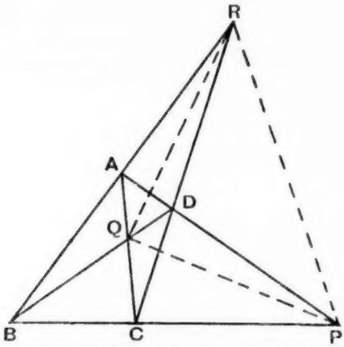
One of the most elementary applications of Archimedes's Principle is the measurement of the specific gravity of a solid by weighing it in air and also when immersed in a liquid of known specific gravity. The " loss in weight " of the solid is equal to the weight of the liquid displaced. Alternatively, the " gain in weight " of the liquid and container when the solid is immersed in the liquid can be used, for it is also equal to the weight of the liquid displaced.

All this is common knowledge and practice, and indeed there is a demonstration well known to teachers of the subject, which shows simultaneously the gain and loss in weight, and also records their equal magnitudes. But it occurs to the writer that it is not widely recognised that this provides an interesting example of action and reaction, which can easily be measured by direct weighings and shown to be equal and opposite. It may therefore be used as a simple, but effective, illustration of Newton's Third Law. This is all the more valuable because of the dearth of such illustrations, even though it suffers from the disadvantage of being a statical example. The difficulty usually is that the action and reaction cannot be isolated, and all one can do is to note their null resultant.

The null resultant can, of course, be recorded in the case mentioned above by weighing the container and the liquid with the solid *freely* immersed. In fact, by the three sets of simple weighings it is possible to measure the action and reaction separately and also jointly. They illuminate not only the third law, but aid the understanding of the nature of fluid pressure. J. TOPPING.

1917. *Note on diagonal triangles.*

Let A, B, C be the triangle of reference, for areal coordinates (x, y, z) where $x+y+z=1$. Let D be a fourth point (l, m, n), and PQR the diagonal triangle of $ABCD$.



AD is the line $y/m=z/n$, and hence P is the point $\{0, m/(m+n), n/(m+n)\}$; similarly for Q and R. Hence the area of PQR

$$=\Delta \begin{vmatrix} 0 & y_P & z_P \\ x_Q & 0 & z_Q \\ x_R & y_R & 0 \end{vmatrix}$$

$$=\Delta \begin{vmatrix} 0 & m & n \\ l & 0 & n \\ l & m & 0 \end{vmatrix} \div (m+n)(n+l)(l+m).$$

where Δ is the area of ABC. Using the relation $l+m+n=1$,

$$\frac{2}{PQR}=\frac{(1-l)(1-m)(1-n)}{lmn\Delta}=\frac{mn+nl+lm-lmn}{lmn\Delta}$$

$$=\frac{1}{l\Delta}+\frac{1}{m\Delta}+\frac{1}{n\Delta}-\frac{1}{\Delta}.$$

But

$$\text{area } DBC=\Delta \begin{vmatrix} l & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=l\Delta=BCD.$$

$$DCA=m\Delta=-CDA,$$
$$DAB=n\,\Delta=DAB,$$
$$ABC=\quad \Delta=ABC,$$

where the third column is cyclic in A, B, C, D. Thus

$$\frac{2}{PQR}=\frac{1}{BCD}-\frac{1}{CDA}+\frac{1}{DAB}-\frac{1}{ABC}.$$

Given four lines a, b, c, d with a diagonal triangle of sides p, q, r, then in a similar manner:

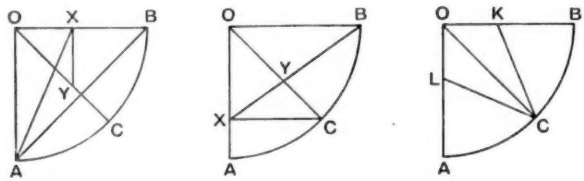
$$\frac{4}{pqr}=\frac{1}{bcd}-\frac{1}{cda}+\frac{1}{dab}-\frac{1}{abc},$$

where bcd is the area of the triangle enclosed by the lines b, c, d.

Is there any simple geometrical proof of these results, and is their similarity due to the principle of duality? It seems surprising, but correct, that one result has 2 where the other has 4. B. I. HAYMAN.

1918. *A School Certificate question.*

The following question was set in a recent School Certificate paper: "A quadrant of a circle is given. Draw a circle to touch the two bounding radii and the arc of the quadrant." Readers may be interested to compare the following constructions, which are easily verified.



1. Join AB, and draw AX, the bisector of $\angle OAB$, to meet OB at X. At X draw XY perpendicular to OB, meeting the bisector OC of $\angle AOB$ at Y. Then Y is the centre of the required circle.

2. Let the bisector of $\angle AOB$ meet the quadrant at C. Draw CX perpendicular to OA to meet OA at X, and join XB to meet OC at Y. Again Y is the required centre.

3. With C as in 2, construct the angles OCK, OCL, each equal to $22\frac{1}{2}°$, L, K being on OA, OB respectively. Then L, K are the points of contact of the required circle with OA, OB. A. B. MAYNE.

1919. *Tests of divisibility by* 7.

1. Multiply the leading digit by 3, and add the next digit; multiply the result by 3, and add the next digit, and so on, subtracting or adding at any stage any multiple of 7. If the final result is a multiple of 7, so is the original number.

Example. 38934.

$3\times 3=9\equiv 2$; $2+8\equiv 3$; $3\times 3=9\equiv 2$;

$2+9\equiv 4$; $3\times 4=12\equiv 5$; $5+3\equiv 1$; $3\times 1=3$;

$3+4=7$.

Thus 38934 is a multiple of 7.

Proof. Let

$$N=a\,.\,10^{k-1}+b\,.\,10^{k-2}+c\,.\,10^{k-3}+\ldots+g\,.\,10^2+h\,.\,10+j.$$

The steps give us:

$$3a+b,$$
$$3^2a+3b+c,$$
$$3^3a+3^2b+3c+d,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

and finally,

$$P = a \,.\, 3^{k-1} + b \,.\, 3^{k-2} + \ldots + 3h + j.$$

Then $\quad N - P = a(10^{k-1} - 3^{k-1}) + b(10^{k-2} - 3^{k-2}) + \ldots + h(10-3).$

When n is an integer, $10^n - 3^n$ has a factor $10-3$.

Hence $$N - P = M(7)$$

and $N - P \pm M(7) = M(7)$; thus if $P = M(7)$, then $N = M(7)$.

This method has the advantage that it is fairly easily applied mentally, and that it will give the remainder when the test number is divided by 7, since it depends on $N - P$ being $M(7)$ and not M(multiple of 7).

2. Multiply the last digit of the number by 5, add the last but one, and multiply by 5, and so on, subtracting $M(7)$ at any stage. If the final result is $M(7)$, so is the original number.

The preceding example gives

$$20 \equiv 6, \; 9 \equiv 2, \; 19 \equiv 5, \; 25 \equiv 4, \; 12 \equiv 5, \; 28 \equiv 0.$$

Proof. With N as above, the steps are

$$5j + h,$$
$$5^2 j + 5h + g,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$P = 5^{k-1} \,.\, j + 5^{k-2} h + \ldots + 5b + a.$$

Thus $\quad 10^{k-1}P = 50^{k-1} \,.\, j + 10 \,.\, 50^{k-2} \,.\, h + \ldots + 10^{k-2} \,.\, 50 \,.\, b + 10^{k-1} \,.\, a$

and $\quad 10^{k-1}P - N = j\{50^{k-1} - 1\} + 10h\{50^{k-2} - 1\} + \ldots 10^{k-2} \,.\, b \,.\, \{50 - 1\}.$

But
$$50^n - 1 = (49+1)^n - 1 = M(49) + 1 - 1$$
$$= M(49) = M(7).$$

Thus $$10^{k-1}P - N \pm M(7) = M(7).$$

Hence if $N = M(7)$, so is P, since 10^{k-1} is not.

3. A method less easily applied mentally, but interesting, is as follows: double the last digit and *subtract* the last but one; double the result and *add* the next digit; and so on, alternately subtracting and adding, and, of course, reducing modulo 7 at any stage. If the result is $M(7)$, so is the original number.

The above example gives the steps 8, 5, 10, $19 \equiv 5$, 10, 2, 4, 7.

Proof. With N as before, the steps give

$$2j - h,$$
$$2^2 j - 2h + g,$$
$$2^3 j - 2^2 h + 2g - f,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P = 2^{k-1} j - 2^{k-2} h + 2^{k-3} g \ldots + (-)^{k-1} a,$$

and $\quad 10^{k-1}P = 20^{k-1} j - 20^{k-2} \,.\, 10h + 20^{k-3} \,.\, 10^2 g \ldots + (-)^{k-1} 10^{k-1} a.$

If k is even, the last term is negative and

$$10^{k-1}P + N = j(20^{k-1} + 1) - 10h(20^{k-2} - 1) + 10^2 g(20^{k-3} + 1) \ldots + 10^{k-2} b(20+1).$$

But
$$20^{2n-1} + 1 = (21-1)^{2n-1} + 1 = M(21) = M(7),$$
$$20^{2n} - 1 = (21-1)^{2n} - 1 = M(21) = M(7).$$

Hence $$10^{k-1}P + N \pm M(7) = M(7).$$

Hence if P is $M(7)$, so is N, since 10^{k-1} is not.

If k is odd, a similar argument applies to $10^{k-1}P - N$.

L. W. Clarke.

1920. ***Tests for divisibility.***

1. Tests for divisibility of an integer by 2, 4 ... 2^ν; 3, 9, 5 and 10 are well known. The test for divisibility by 7, 11 or 13 has been pointed out by Hardy and Wright.*

In this test the property that $7 \times 11 \times 13 = 1001$ is made use of.

We can extend this process and get a few more tests for divisibility.

2. It is easily seen that

$$10^r + 1 \equiv 0 \pmod{p},$$

when

$$\begin{array}{lll} r=1 & \text{and} & p=11, \\ r=2 & \text{and} & p=101, \\ r=3 & \text{and} & p=7,\ 11 \text{ or } 13, \\ r=4 & \text{and} & p=73 \text{ or } 137. \end{array}$$

Hence to test whether a number is divisible by any of these numbers, the rule is :

" Form blocks of r-digit numbers from the right end (unit digit). (The extreme left block may or may not have all the r-digits.) Then find the result got by alternate block addition and subtraction. If the number got by this process is divisible by p, the given number is also divisible by p."

E.g. To test divisibility by 73 or 137 :

Consider the number 837, 362, 172, 504, 831. Form blocks of 4 digit numbers and take alternate addition and subtraction :

$$837 \mid 3621 \mid 7250 \mid 4831.$$

$$\begin{array}{rr} 837 & 3621 \\ \underline{7250} & \underline{4831} \\ 8087 & 8452 \\ & \underline{8087} \\ & 365 \end{array}$$

Since 365 is divisible by 73, the given number is divisible by 73 but not by 137.

3. Mr. D. R. Kaprekar † has given a test for divisibility by 37. The rule is : " Form blocks of three digit numbers from the right end. If the sum of these three digit numbers is divisible by 37, the number is divisible by 37."

In this test the property that " 999 is a multiple of 37 " is made use of.

NOTE.—111 is also a multiple of 37.

4. This can again be extended by solving the equation $10^r - 1 \equiv 0 \pmod{p}$.

NOTE.—*Since* $(10^r - 1)$ *can be factorised when* r *is even and* $10^r + 1 \equiv 0 \pmod{p}$ *has been considered in* § 2, *it is enough if we give values to* r, *numbers which are odd.*

Thus

$$\begin{array}{lll} r=1 & \text{and} & p=3, \\ r=3 & \text{and} & p=37, \\ r=5 & \text{and} & p=41,\ 271, \\ r=7 & \text{and} & p=239,\ 4649. \end{array}$$

* G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers* (Oxford, 1938).

† D. R. Kaprekar, *Ten Cuts in Calculation* (India, 1935).

Hence the rule: "Form blocks of r-digit numbers from the right-hand end. If the sum of these blocks of numbers is divisible by p, then the number is divisible by p."

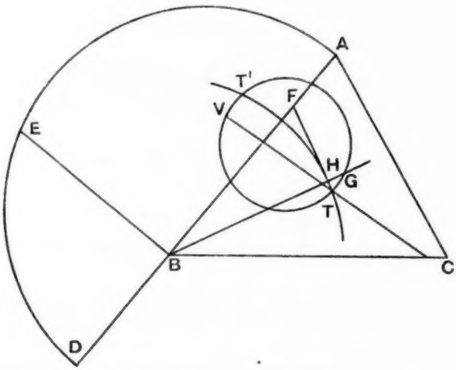
Note that 11111 ... (r digits) is also a multiple of p.

These results are not well known, though probably not new. The absence from Hardy and Wright of the test for divisibility by 37, a test given by Kaprekar in his book in 1935, prompted me to write this note.

S. PARAMESWARAN.

1921. *On Note* 1730 (XXVII, p. 112).

Another construction for bisecting a triangle by a line through a given point is shown in the figure; it depends on the equal-area property of the tangents of a hyperbola.



In the figure, $BD = \frac{1}{2}BC$, a semicircle is drawn on AD as diameter and BE is perpendicular to BA. The line BG bisects the angle ABC and

$$BF = BG = BE.$$

The line FH is at right angles to BG. The circle with VG as diameter is drawn and cuts the circular arc whose centre is B and radius BH in T and T'. Then VT or VT' will bisect the triangle ABC; in the figure VT is the bisector. The focus of the hyperbola is G, its vertex H, its asymptotes are BA and BC, and VT touches it.

J. H. CADWELL.

1922. *Porism of the hexagon.*

In Casey's *Sequel to Euclid*, p. 158, ex. 89, the condition is given that two circles should be such that a hexagon can be inscribed in one and circumscribed to the other.

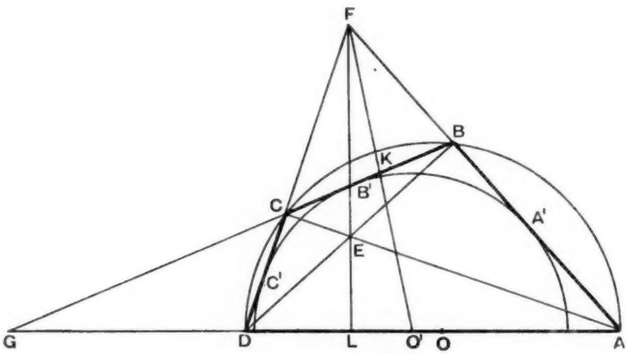
The form of the condition indicates the method of proof. The joins of alternate vertices also touch a coaxal circle, whose radius and centre are given by exs. 57 and 58 on p. 154. The last circle is then such that a triangle can be circumscribed to it which is inscribed in the first circle. Thus these two circles satisfy a well-known condition from which the condition of ex. 89 can be deduced.

A different method of approach to the problem is given in an article on *Poncelet's Poristic Polygons*, by F. V. H. Gulasekharam, in the *Math. Gazette*, Feb. 1941, p. 28.

The condition, however, is somewhat complicated, and the following theorem exhibits the relation between the two circles in a much simpler form.

Let two circles with radii R, R' and centres O, O' ($OO'=d$, and $R'+d<R$) be such that a hexagon can be inscribed in the former and circumscribed to the latter circle. Let A, D be the ends of the common diameter on the outer circle, and AB, DC chords of it touching the inner circle.

Then $AB+DC=AD$.



This result is easily proved trigonometrically, but the geometrical proof brings to light a number of interesting properties of the figure, which is half of that obtained when the hexagon is taken symmetrical about the common diameter. From the poristic property, it is evident that BC also touches the circle centre O'.

Suppose AB, BC, CD touch the circle (O') at A', B', C'. Let AC, BD; AB, DC ; AD, BC meet in E, F, G. $O'F$ meets BC at K, FE meets AD at L.

Then :

(1) *G, L are limiting points of the two circles.*

For $AC \perp DC$, $DB \perp AB$, hence E is the orthocentre of the triangle FDA, and hence $FE \perp AD$.

Since EFG is the diagonal triangle of the cyclic quadrangle $ADCB$, EF is the polar of G for the circle (O). Also $F(GCEB)$ is harmonic ; hence as FC, FB are tangents to the circle (O'), FG, FE are conjugate lines for the circle (O'). Hence the pole of FE for the circle (O') lies on FG, and so must be G, since $FE \perp AD$.

Thus the pole of EF for both circles is G, which is therefore one limiting point ; and hence L is the other.

It should be observed that since CLB is the pedal triangle of AFD, CL and BL are equally inclined to AD, and therefore meet the circle (O) again at B_1, C_1, the reflections of B, C in AD. Thus the joins of the opposite vertices of the hexagon $ABCDC_1B_1$ pass through a limiting point. This is true for all hexagons inscribed and circumscribed to the two circles, as is shown in the article by F. V. H. Gulasekharam cited above, p. 31.

(2) *$A'C'$ passes through G.*

For $A'C'$ is the polar of F for the circle (O').

(3) *$A'C'$ bisects OK at right angles.*

For the bisectors of the angles formed by pairs of opposite sides of the cyclic quadrangle $ABCD$ are perpendicular; hence as GA' is perpendicular to the bisector of $\angle AFD$, *i.e.* to $O'F$, it must bisect $\angle AGB$. Hence:

(4) *$O'A'KC'$ is a rhombus.*

All its sides $= O'A' = R'$.

(5) *$A'K$ is parallel to AC.*

For $A'K$ is parallel to $O'C'$ which is perpendicular to CD, and therefore parallel to AC.

(6) $\cos A + \cos D = 1$, where $A = \angle DAB$, $D = \angle ADC$.

For projecting $O'A'$, $A'K$ orthogonally on $O'B'$, it follows that the sum of the projections of $O'A'$, $O'C'$ on $O'B' = O'B'$. Also $\angle B'O'A' = 180° - B = D$, $\angle C'O'B' = 180° - C = A$.

Thus $R' \cos D + R' \cos A = R'$, giving the result.

Hence, finally:

(7) $AB + DC = AD$, by multiplying (6) through by $2R$.

The condition in Casey may now be obtained by writing (7) in the form

$$\frac{1}{\cos A \cos D} = \frac{1}{\cos A} + \frac{1}{\cos D},$$

and noting that $$\sin A = \frac{R'}{R+d}, \quad \sin D = \frac{R'}{R-d}.$$

It may also be seen that conversely if two points B, C are taken on the semicircle with diameter AD such that $AB + DC = AD$, the circle which has its centre O' on AD and touches AB, CD will also touch BC, and the two circles (O), (O') are such that a hexagon can be inscribed in the former whose sides touch the latter circle.

H. V. MALLISON.

1923. *A further note on the circle of curvature.*

In my note on the circle of curvature in cartesian coordinates (*Math. Gazette*, Dec., 1944, p. 188), based on the expansion for a group of order p,

$$y = x(a + a_1 x^{\frac{1}{p}} + a_2 x^{\frac{2}{p}} + \ldots + a_{p-1} x^{\frac{p-1}{p}} + bx + \ldots),$$

there is an unfortunate omission and a somewhat misleading statement in the too brief paragraph dealing with the case in which $a = +\iota$ or $-\iota$.

Instead of the limited form there given, we can consider equally well, and with the same method, the group of branches through a point P on the curve which are given by

$$y = x(\iota + d x^{\frac{s}{p}} + \ldots),$$

where $1 \leqslant s$ and d is not zero but s may exceed p, leading to the family of degenerate circles for which $r = p + s$ (instead of $p + 1$ as stated there and earlier in the note), and a circle of curvature with $2p + s$ intersections at P. In general, the point P will be imaginary, involving a translation of the axes to an imaginary point, and referred to the original axes the equation of the circle of curvature will be of the form

$$(x - l - \iota m)^2 + (y - l' - \iota m')^2 = 0.$$

This circle has two real points on it, $(l - m', l' + m)$ and $(l + m', l' - m)$, which, assuming the original equation has real coefficients and since the tangent at P is a circular line, are a focus of the curve and its image in the corresponding

directrix, as will readily be seen from the figure obtained by a generalised projection. It is therefore better described as a circle of zero radius than as a point circle. When P is real (as at a real circular isolated point) these points coincide with P, and the circles of the two conjugate branches coalesce, having in all $4p+2s$ intersections at P with them. The investigation includes the interesting possibility of $1+a^2$ and b both zero, when $(1+a^2)/b$ of our general but here inapplicable formula becomes indeterminate.

In the next paragraph of the earlier note it was tacitly assumed that $1+a^2$ was not zero.

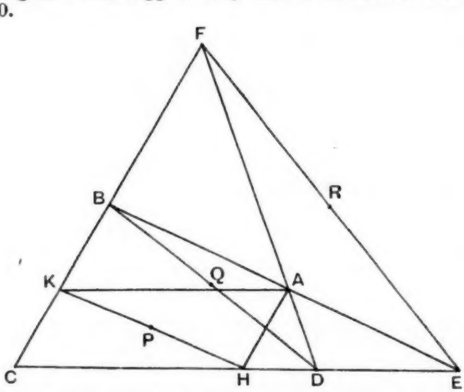
The whole matter may be summarised thus: With our definition of the circle of curvature, there is a circle of curvature at a point P of the curve through which passes a group of branches of order p if the $p+1$th approximation to the group is parabolic (including the degenerate case of the tangent and the line at infinity, and the radius of curvature is then infinite unless the tangent at P is a circular line when it is zero): if the approximation is not parabolic, the circle of curvature at a point Q of a branch tends to a circle of zero radius as Q approaches P, but this limiting circle (the circular lines through P) is not a circle of the system unless the tangent at P is a circular line.

Some limitations to the values of p and s are at once available. We will suppose the curve is of degree n greater than 2 and in no way degenerate. For the existence of the circle of curvature in the general case of non-zero radius we must have $p<n/2$: in the special case of zero radius, however (with the coefficients of the curve's equation referred to real axes assumed real), if the point P and its conjugate P' are distinct, $p \leqslant n/2$ and $p+s<n$, but if P and P' coincide, $2p+s \leqslant n$.

H. Gwynedd Green.

1924. *The diagonals of a complete quadrilateral.*

The following note was suggested by Mr. W. J. Dobbs' Note 1110, *Gazette*, XVIII, p. 200.



1. In the figure, $ABCDEF$ is a complete quadrilateral; AH is parallel to BC, and AK to DC. Then

$$\begin{aligned} HD/HE &= (HD/HA)\,.\,(HA/HE) \\ &= (KA/KF)\,.\,(KB/KA) \\ &= KB/KF \\ &= m_1/(m_1+m_2), \text{ say.} \end{aligned}$$

2. Place particles m_2 at H and K, and particles m_1 at E and F. Let P, Q, R be the midpoints of HK, DB, EF respectively. Then m_2 at H and K are equivalent to $2m_2$ at P; and m_1 at E and F are equivalent to $2m_1$ at R. Hence the centroid of the four particles lies on PR. Also m_2 at H and m_1 at E are equivalent to m_1+m_2 at D, while m_2 at K and m_1 at F are equivalent to m_1+m_2 at B. Hence the centroid of the four particles is at Q. Hence P, Q, R are collinear and $PQ : QR = m_1 : m_2$. Also the midpoint of HK is that of AC. Hence the midpoints of the diagonals of a complete quadrilateral are collinear.

3. In the figure $AE/AB = HE/AK$, and $AF/AD = AK/HD$.

Hence $$AE \,.\, AF/AB \,.\, AD = HE/HD = (m_1+m_2)/m_1 \text{ by } 1.$$

Hence $$m_1 \,.\, AE \,.\, AF = (m_1+m_2)AB \,.\, AD,$$

or $$m_1(AE^2+AF^2-EF^2) = (m_1+m_2)(AB^2+AD^2-BD^2).$$

4. By the standard formula for two particles and their centroid, *i.e.* in this case, for $2m_2$ at P, $2m_1$ at R, $2(m_1+m_2)$ at Q,

$$m_2AP^2+m_1AR^2 = (m_1+m_2)AQ^2 + m_1m_2PR^2/(m_1+m_2)\,;$$

i.e. $$\tfrac{1}{4}m_2AC^2 + \tfrac{1}{2}m_1(AE^2+AF^2-\tfrac{1}{2}EF^2)$$
$$= \tfrac{1}{2}(m_1+m_2)(AB^2+AD^2-\tfrac{1}{2}BD^2) + m_1m_2PR^2/(m_1+m_2).$$

Using equation 3 this becomes

$$\tfrac{1}{4}m_2AC^2 + \tfrac{1}{4}m_1EF^2 = \tfrac{1}{4}(m_1+m_2)BD^2 + m_1m_2PR^2/(m_1+m_2).$$

But from 2, $$m_1 : PQ = m_2 : QR = (m_1+m_2) : PR.$$

Hence $$\tfrac{1}{4}QR \,.\, AC^2 + \tfrac{1}{4}PQ \,.\, EF^2 + \tfrac{1}{4}RP \,.\, BD^2 + PQ \,.\, QR \,.\, RP = 0.$$

Hence the circles having the three diagonals of the quadrilateral as diameters are co-axial.

N. M. GIBBINS.

1925. *Approximations to roots.*

Are we getting tired yet of approximations to roots? If not, the following may be of interest on account of its simplicity and effectiveness.

When teaching elementary progressions and the associated means I generally observe that the geometric mean of two numbers is also the G.M. of their arithmetic and harmonic means, and that as these numbers are considerably closer together than the original numbers we can by repeating the process approach the G.M. simultaneously from both sides, averaging the final pair (which is, of course, the same as finding one more A.M.) when we have had enough. *E.g.* to find $\sqrt{6}$, since $6 = 3 \times 2$ we have:

(1) A.M. $= 2{\cdot}5$,
H.M. $= 2{\cdot}4$.
(2) A.M. $= 2{\cdot}45$,
H.M. $= 2{\cdot}448979592-$.
(3) A.M. $= 2{\cdot}449489796-$,

where I have purposely carried the computation to a rather alarming length for comparison with the correct value,

$$2{\cdot}449489743-.$$

The agreement, already to the 8th figure, is startling. On investigation, we find that the theoretical agreement is, in fact, closer than we might reasonably have expected, thus: let A, G, H be the arithmetic, geometric, and harmonic means of $1+h$ and 1.

Then : $$A = 1 + \tfrac{1}{2}h,$$

$$H = \frac{2(1+h)}{2+h} = 1 + \tfrac{1}{2}h(1+\tfrac{1}{2}h)^{-1}$$

$$= 1 + \tfrac{1}{2}h - \tfrac{1}{4}h^2 + \tfrac{1}{8}h^3 - \tfrac{1}{16}h^4 + \dots ;$$

thus $$\tfrac{1}{2}(A+H) = 1 + \tfrac{1}{2}h - \tfrac{1}{8}h^2 + \tfrac{1}{16}h^3 - \tfrac{1}{32}h^4 + \dots .$$

Now $$G = (1+h)^{\frac{1}{2}}$$

$$= 1 + \tfrac{1}{2}h - \tfrac{1}{8}h^2 + \tfrac{1}{16}h^3 - \tfrac{5}{128}h^4 + \dots .$$

Hence $$\tfrac{1}{2}(A+H) - G = \tfrac{1}{128}h^4 \text{ approx.}$$

Replacing $1+h$ and 1 by a and b $(a > b)$, the error can be written :

$$(a-b)^4/128b^3.$$

With $a = 2{\cdot}5$, $b = 2{\cdot}4$ this is approximately 5×10^{-8}, which enables us to deduce $\sqrt{6} = 2{\cdot}4494897$ correct to 8 significant figures.

Of course, the method works best for conveniently chosen numbers (but what method doesn't ; don't H.C. examiners always set $\sqrt{26}$ and $\sqrt[4]{80}$ as exercises on the Binomial Series?), and soon becomes unwieldy, but if good factors are chosen for the start, one application may give us all the accuracy we want. *E.g.* to find $\sqrt{3}$, since $3 \times 16 = 49$ nearly, $\sqrt{3} = 7/4$ nearly. Thus, take $3 = \frac{7}{4} \times \frac{12}{7}$; with very little labour we obtain :

$$\text{error} = 2{\cdot}5 \times 10^{-9} \text{ approx.}$$

$$A = \tfrac{97}{56} = 1{\cdot}7321428571 + ,$$

$$H = \tfrac{168}{97} = 1{\cdot}7319587629 - .$$

Hence $$\tfrac{1}{2}(A+H) - \text{error} = 1{\cdot}7320508075,$$

whence $\sqrt{3} = 1{\cdot}73205081$, correct to 9 significant figures. (The above result is actually correct to 10.)

Even a fairly crude beginning such as $20 = 4 \times 5$ gives us $\sqrt{20} = 4{\cdot}4721$, which is correct to 5 significant figures, very easily, in one go. Perhaps the 3rd Form method would be *almost* as easy in this instance—but much less fun!

The method is, of course, only a disguised form of the familiar Newton iteration, that if r is an approximation to $\sqrt{x}$, a better approximation is $\frac{1}{2}(r + x/r)$.

A. R. PARGETER.

1926. *Preparing teaching time-tables.*

Referring to the article on page 176 of Volume XXVIII, these notes arise from methods used in a school of 26 to 28 staff in restricted accommodation. They assume a 4-period morning and a 3-period afternoon.

§ 5 (3) says : " Work with the teacher as the unit . . . ' map-marker '." It is the class-subject relation which is constant during the year : the teacher-class relation (or even the teacher-subject relation) is not necessarily constant, especially in wartime. Again, we may have two small classes under one teacher (*e.g.* forms VI P.T.). At a time when four temporary teachers left at once and were replaced by four others who covered the same class subjects collectively but not individually, I decided *in construction* to change from Table V (showing for the class the teacher-subject in each period, on a movable flag) to a teacher-period grid with the *class* on a flag. The time of construction was found in later full-scale time-table-making to be reduced by over 50 p.c., and rapid adjustment to staff changes became almost easy.

The construction board is ruled :

Staff	*Period*		
	1	2	3
⋮			
c			
b			
a			

(reading staff upwards so that a turn of the board through 270° gives a replica of the published staff time-table). The class-flag bears indication of the subject.

The advantages of this are :

(1) 35 flags are cut for a 35-period week, and coloured to represent the class. This is self-checking.

(2) The class keeps the same distribution of subject-periods, and for change of teacher-class the class flags move from one horizontal line to another, often without horizontal translation (see (3)).

(3) Marking and special non-teaching periods are distributable at will, that is, there is no need to find the teacher's marking time all on one day.

(4) The procedure for split and set forms is greatly simplified and answers the qualification of p. 180, § 5 (4). The removal of the left free corner of a rectangular flag (on a pin) shows a part-form and is an instant warning not to move this flag without moving that of the other part. For mixed classes remove the left corner for boys, the right for girls, and for classes split regardless of sex remove the middle of the base. Thus p. 180, § 5 (4) (*e*) is dealt with.

(5) After the double period and special cases have been dealt with, drop in the teacher's line for the day his class-subject flags. When all flags have thus been assigned (but not arranged), begin at the top of *one day* and work downwards until a consistent distribution is obtained for that day. Usually very few solutions are possible, but the presence of blanks in the lines gives wide play to human considerations (*e.g.* teacher *a* need not have class *A* last period every afternoon).

NOTES. 1. The problem of construction is complicated by two (or more) buildings at a distance. Free allocation of class subjects to teachers becomes difficult, and " rings " of teacher-class-periods must be sought, so that, say, 6 classes in the remote building have 6 teachers permuted for the 4 or 3 lessons in a session.

2. Rooms in restricted accommodation are dealt with again by the self-checking device of 35 numbered discs pierced to be held in place by the class flag-pins. A spare horizontal line holds spare room-discs (if any) to deal with temporary dislocations such as burst pipes or falling ceilings. Colours warn the constructor of specialist or small rooms.

3. Referring to (5) above, how can the number of possible Monday (say) time-tables be worked out?

J. C. E. WREN.

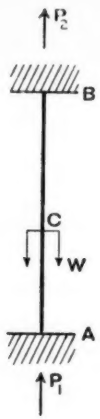
1927. *Column axially loaded between supports.*

Consider a strut AB, subject to an axial load W at C.

Let $BC=b$, $CA=a$; let A be the area of the column and E the modulus of elasticity ; λ_1, λ_2 the stiffnesses of the supports at A and B ; P_1, P_2 the vertical reactions at A and B. Then

$$P_1+P_2=W, \qquad \text{(i)}$$

$$\frac{P_2}{\lambda_2}-\frac{P_1}{\lambda_1}=-\frac{P_2 b}{EA}+\frac{P_1 a}{EA}. \qquad \text{(ii)}$$



From (ii),

$$P_1 : P_2 = \left(\frac{1}{\lambda_2} + \frac{b}{EA}\right) : \left(\frac{1}{\lambda_1} + \frac{a}{EA}\right) = k, \text{ say,} \qquad \text{(iii)}$$

and then from (i):

$$P_1 = W/(1 + 1/k), \quad P_2 = W/(1+k),$$

where k is given by (iii).

If λ_1, λ_2 are infinite, $k = b/a$ and

$$P_1 = bW/(a+b), \quad P_2 = aW/(a+b).$$

Frequency of longitudinal vibration.

Let the deflection of the load from its equilibrium position be x, then

$$\frac{W}{g}\frac{d^2x}{dt^2} + F = 0,$$

where F is the "out of balance" force, and so

$$F = x\left[\frac{1}{\dfrac{b}{EA} + \dfrac{1}{\lambda_2}} + \frac{1}{\dfrac{a}{EA} + \dfrac{1}{\lambda_1}}\right] = \mu x. \qquad \text{(iv)}$$

Hence the frequency of vibration is

$$\frac{1}{2\pi}\sqrt{\frac{g\mu}{W}},$$

where μ is given by (iv).

If λ_1, λ_2 are infinite, $\mu = EA(a+b)/ab$, and the frequency of vibration is

$$\frac{1}{2\pi}\sqrt{\left\{\frac{gEA(a+b)}{Wab}\right\}}.$$

D. W. J. Cruickshank.

1494. Halfpenny Bridge at Lechlade gets its name from the toll that was once levied there. An interesting feature of its construction is that the keystone is not bevelled and depends entirely on friction to perform its duties.—Robert Gibbings, *Sweet Thames Run Softly*, pp. 12–13. [Per Mr. F. W. Kellaway.]

VICTORIA BRANCH.

Report for 1945.

We regret to record the deaths of two of our Vice-Presidents: Mr. R. J. A. Barnard and Dr. J. M. Baldwin, both of whom had been members of the Association for many years. Mr. Barnard was the leading spirit in the formation of the Branch in 1906. He was its secretary from this time until 1934, and then President until 1938. The Association has had no member more active and enthusiastic. Dr. Baldwin, who was Government Astronomer for Victoria, joined the Association in 1909; he was elected a vice-president in 1924 and retained this office until his death.

We accepted, with regret, the resignation of our president, Mr. D. K. Picken, who is now living in the country. Mr. Picken has been president for the past seven years and, despite the effect of the war, has seen the Association grow during his term of office. Mr. Picken has accepted the position of Vice-President to which he was unanimously elected.

During the year we held five meetings as follows:

23rd March. Annual Meeting. Professor Cherry elected president. Associate Professor Belz spoke on "Statistics in relation to Matriculation examination". The audience were keenly interested in the subject and a good discussion followed.

27th April. Professor Cherry gave an address on "Rockets" to a combined meeting of the Mathematical Association, University Mathematical Society and an Inter-Planetary Society recently formed by a group of student enthusiasts. The attendance at this meeting broke all records. Much of the calculation was directed towards the possibility of a "return" trip to the moon.

22nd June. Mr. H. B. Sarjeant gave a very interesting paper on "Mathematics in the Field of Sport". He showed how a knowledge of trigonometry and dynamics might *perhaps* lead to a reduction in one's golf handicap.

20th July. Mr. Picken gave a postponed presidential address on "Mathematics as Culture". The speaker described himself as fascinated by the perfection of mathematics: the only realm of absolute perfection in human knowledge—making it of the utmost philosophical importance. Supreme beauty is the ultimate test of true mathematics.

28th September. Dr. F. A. Behrend broke new ground when he gave an interesting lecture "On constructions with ruler and unit measure". Erection of perpendiculars, bisection of angles, construction of parallel lines and even a regular polygon of seventeen sides were achieved.

This has been our second year of activity after closing down owing to pressure of R.A.A.F. work. We have been gratified by the increasing numbers of members and associates, as well as by the fact that 156 Victorians have subscribed to the *Australian Mathematics Teacher* which is published by the N.S.W. branch.

We hope to welcome back S/L. F. J. D. Syer in 1946 and look forward to a very prosperous year.

T. M. Cherry, *President.*
H. B. Sarjeant, *Secretary.*

SOUTHAMPTON BRANCH.

On 30th November, Mr. Tuckey read a paper entitled "Teachers and Examiners". On 8th February, Miss N. Walls read a paper on "Adjustable Cycloids", giving a resumé of Morley's work, and continuing with further developments of her own.

Mr. Tuckey's paper on 1st March on "The Teaching of Geometry under the new regulations" produced a lively discussion.

Future meetings have been arranged as follows :

14th June. Professor Neville : "Mathematics can't do sums".

25th October. Mathematical Films, introduced by Mr. I. R. Vesselo. Other speakers will include Mr. Robson, Mr. Roseveare, and Dr. Trubridge.

There are now 22 full members, 11 associates and 14 student members, and the secretary would be glad to hear of any other potential members in the district.

F. G. MAUNSELL, *Hon. Secretary.*

University College, Southampton.

LONDON BRANCH.

On 18th May, a joint meeting of the Branch and the Scientific Film Association saw a display of Mathematical Films. There was an attendance of 120, who were very interested in the viewpoints of two film producers, Mr. Fairthorne and Mr. Segaller. A lively discussion followed, in which a number of differing views on the future of the film were expressed.

A. J. G. MAY, *Hon. Secretary.*

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

DEAR SIR,—May we use your columns to make the following suggestions to teachers ? They deal with rather trivial matters, but we think that their adoption would save some time and make for clearness.

(i) The word "isosceles" should not be used. See *Gazette* XXX, p. 35.

(ii) In expressions like $x+y\times z$ brackets should be included to show whether $(x+y)\times z$ or $x+(y\times z)$ is intended. Although the convention "multiplication before addition" is often assumed, it is not a useful convention. In fact $x+y\times z$ rarely occurs outside elementary examination papers. Elsewhere it is usually $(x+y)z$ or $x+yz$.

(iii) The conventional distinction between $\sqrt{x}$ and $x^{\frac{1}{2}}$ according to which the first denotes the positive square root and the second carries the ambiguous sign ought to be accepted. It is in fact used at the stage of quadratic equations when the solution of $ax^2+2bx+c=0$ is generally expressed in terms of $\pm\sqrt{(b^2-ac)}$. It is useful to adhere to it later, especially in integration, and even to extend it in complex algebra so that for example $\sqrt[3]{8}$ means 2, but $8^{\frac{1}{3}}$ is three-valued. Possibly some of the following items (iv to vii) require further consideration.

(iv) At a later stage straight lines and curved lines are always called lines and curves respectively. Could this be done from the start ?

(v) The quadrangle is a figure composed of four points and the quadrilateral a figure composed of four lines. Thus for example four points on a circle form a cyclic quadrangle, not a cyclic quadrilateral. Could this also be adopted in elementary courses ? The word trilateral might sometimes be used with advantage, *e.g.* in referring to the inscribed circle.

(vi) The short reasons SAS, SSS, AAS, RHS for congruence are now in very general use. Would it be possible to agree on abbreviations for the corresponding tests of similarity ?

(vii) In elementary work, is it better to use AB to denote the whole line through A and B, or only the part of it from A to B ? Can we dispense with such phrases as "on AB or AB produced or on BA produced", or do they help the beginner ?

Yours etc., C. V. DURELL.
A. ROBSON.

REVIEWS.

An index of mathematical tables. By A. Fletcher, J. C. P. Miller and L. Rosenhead. Pp. viii, 451. 75s. 1946. (Scientific Computing Service)

No one who receives frequent requests for information about mathematical tables can be unaware of the difficulties these requests may raise: wanted, an eight-figure table, does one exist, is it reliable, is it convenient to use, is it readily obtainable. The last question is one for which library lists may suffice, provided the former questions can be answered, but for these, up to the present, there has been no systematic means of finding the answer. The list in the *Monthly Notices of the Royal Astronomical Society*, prepared and annotated by Dr. Comrie, is useful for the more common functions, and the new American journal, *Mathematical Tables and other Aids to Computation*, is proving of great value. But there has been an obvious need for an index, full, systematically arranged in a single volume, and thoroughly reliable. This we now have in the present admirable work, prepared by a trio of enthusiastic and well-informed writers, members of the staff of the Mathematics Department of the University of Liverpool. The authorship alone would have guaranteed a valuable book; but a happy collaboration has enlisted as publisher the Scientific Computing Service, whose managing director, Dr. L. J. Comrie, stands without a superior, almost without a peer, in his knowledge of everything pertaining to mathematical table-making and computing. As Dr. Comrie is also an enthusiast for typography, his co-operation has contributed to the excellent appearance of the pages,* while a word of praise must certainly be given to the printers, Messrs. Neill of Edinburgh; their share of the work must have been arduous and it has been most skilfully performed.

There are certain obvious questions about the *Index* to which some answer should be made in a review, although only steady, day-by-day routine use can provide material for a final judgment.

First, is the *Index* adequate? It must be clearly understood that the compilers have wisely refrained from aiming at bibliographical completeness; this would be almost impossible to attain, and is of secondary importance in a work which is primarily intended for table-users and computers. Thus the authors tell us that they have rejected, for one reason and another, about 1000 of the 3000 items which have come under consideration. It would therefore be unwise to assume that an omission means an oversight, and there is no evidence of any lack of comprehensiveness.

Secondly, is the *Index* reliable? The names of the compilers and publisher should be a sufficient guarantee. In addition, Professor R. C. Archibald remarks† that he has himself checked hundreds of entries without finding any errors worth noting. Moreover, with an honest caution which wins confidence, listed items which have not been personally inspected by the compilers are marked with an asterisk; those who know how difficult it can be to inspect personally some not particularly rare volume will applaud the thoroughness which has kept the stars down to about 300 in 2000 entries.

Thirdly, is the *Index* conveniently arranged, easy to use, and helpfully annotated? It is probably desirable here to enter into a more detailed account of the arrangement of the contents. The 17 introductory pages explain the

* Even a faddist can find little to cavil at: there are a few "rules" in solid text which could have been avoided; the splendid old-style numeral fount is both beautiful and clear, but its zero is not suitable as a functional argument. These are minutiae hardly proper to be noticed in a review; but most of us know that the authors and the publisher are themselves stern critics of minutiae.

† *Mathematical Tables and other Aids to Computation*, II, 13, p. 18 (January 1946).

principles on which the book was planned, the way to use it, and the symbols employed. This should be read, not only in order to learn how to use the index sections efficiently, but also for much interesting information contained therein.

The main index consists of two parts, Part I the index according to functions, Part II the bibliographical index. Part I is divided into 24 sections, and it may be worth while giving a brief indication of the section contents ; an " etc." covering related topics is to be understood after each item, since the following abbreviations are rather brutal : 1. Primes and factors ; 2. Powers ; 3. Factorials ; 4. Bernoulli and Euler numbers ; 5. Mathematical constants ; 6. Common logarithms ; 7. Natural trigonometric functions ; 8. Logarithms of trigonometric functions ; 9. Inverse circular functions ; 10. Natural and logarithmic exponential and hyperbolic functions ; 11. Natural logarithms of numbers and of trigonometric functions ; 12. Combinations of circular and hyperbolic functions, and circular and hyperbolic functions of a complex variable ; 13. The exponential integral ; 14. The Gamma function ; 15. The error integral ; 16. Legendre functions ; 17, 18, 19, 20. Bessel functions (real argument, imaginary argument, complex argument, miscellaneous) ; 21. Elliptic functions ; 22. Miscellaneous functions of advanced mathematics ; 23. Interpolation, numerical differentiation and integration ; 24. Harmonic analysis.

Each section is divided into decimally-indicated sub-sections. Now, for instance, if we want a 10-figure table of $\log(\sinh x/x)$, we look at section 10, find our way to 10·54, and find four entries ; there is no 10-figure table, but there is, for example :

12 dec. 0(·001)·506 Δ^3 **Pernot & Woods 1918.**

We thus have the number of decimals, the range and interval of tabulation, the information that first, second and third differences are tabulated ; and then a reference to the alphabetically arranged Part II shows that what we have to look for is F. E. Pernot and B. M. Woods, *Logarithms of hyperbolic functions to twelve significant figures*, *Univ. Calif. Publ. in Eng.*, **1** (13), 297–467, published in 1918. The compilers have been particularly careful to supply information about the interpolatory possibilities of a table, by means of about 40 groups of symbols available for their third column.

A few examples will show how thorough is their system :

Δ^2 first and second differences ;
δ^2 second differences only ;
v first variation, (interval times first derivative) ;
PP proportional parts giving every tenth of some differences, rounded to nearest integer ;
fPP full proportional parts, all tenths of all differences, rounded to nearest integer ;
$fPPd$ as for fPP, except that tenths are exact and so given to one decimal.

In the fourth column the use of clarendon, as in the example above, indicates a table which, granted its accuracy, is convenient to use or likely to be readily accessible. Much valuable information is also given in the introductions to each main section, and in the comments on sub-sections or on individual items therein.

About Part II, less need be said, save that it contains a number of references to tables still in manuscript, mentioned of course in Part I. This should help the intending table-maker to avoid duplication of effort. Bibliographically, it might have been helpful to have slightly extended Part II, by supplying the title or subject of items published in periodicals. Thus, for example, " Milne-Thomson 1931*a*, *Proc. London Math. Soc.*, (2) **33**, 160–164 " might have been extended to remind us that it is a " Ten-figure table of the complete elliptic

integrals K, K', E, E', and a table of $1/\vartheta_3^2(0 \mid \tau)$, $1/\vartheta_3^2(0 \mid \tau')$ ". But the compilers will not feel this as a reproach, since it is clear that the user of tables should find the present arrangement and scope of Parts I and II wholly adequate.

All who use mathematical tables must ensure that this volume is readily accessible ; it is an absolutely indispensable adjunct to any library, public or private, of scientific pretensions. Our most grateful thanks go to the authors, who have not allowed the strains of war, the severe pressure of additional academic and national duties, the increased difficulties of all kinds, to prevent them from carrying their laborious task to a most successful and triumphant conclusion.

T. A. A. B.

Advanced Mathematics for Technical Students. I. By A. GEARY, H. V. LOWRY and H. A. HAYDEN. Pp. viii, 419. 12s. 6d. 1946. (Longmans)

The title of this book is perhaps a little misleading, in that it conjures up such ideas as operational calculus, vector methods and harmonic analysis (to quote only a few examples), none of which occur in the subject-matter discussed within its pages. It is only fair to the authors, however, to point out that their use of the word " advanced " is fully explained in the preface and is purely relative, to enable them, in fact, to distinguish between this work and their previous well-known one entitled *Mathematics for Technical Students*, to which it may be regarded as a sequel.

The book is designed primarily for the use of students in science and engineering who have reached Intermediate standard, and covers the ground required for the first year of a final course for an engineering degree. Although any such reader would certainly be acquainted with the elements of the calculus, no previous knowledge is assumed and it is recommended, one feels wisely, that the chapters dealing with differentiation and integration should be read as a whole, in order to supply continuity of thought and the necessary detail usually omitted from a more elementary book.

The authors have very sensibly refrained from over-emphasising the need of pure mathematical rigour, too often introduced at the expense of valuable material in works of this character. Not that one wishes to imply that the treatment is in any way crude or distasteful; in actual fact much care has been taken to give notes and references for the use of students who are desirous of pursuing the finer details.

The book is divided into twelve chapters, dealing, in addition to the calculus already discussed above, with : limits ; infinite series ; the analytical geometry of the straight line, circle and conic sections ; polar coordinates ; volumes and surface areas ; the elements of complex numbers and vector diagrams ; theory of equations ; approximate methods ; and the solution of simple differential equations. Each chapter is provided with a wealth of examples and there are many valuable illustrations, mostly of a practical nature, worked out in the text.

By way of criticism, one feels, perhaps inevitably, that some of the proofs could have been shortened or even omitted, and the order of presentation appears a little peculiar in places ; for example, in Chapter IV (on the straight line and circle) we find a reference to the parametric representation of the parabola which would be more appropriate in the later chapter on conic sections. In this latter chapter the absence of any reference to determinants in dealing with the general equation of the second degree makes the methods employed seem rather artificial ; but determinants are reserved for treatment in the promised second volume.

The book occupying some 400 pages is neatly bound and clearly printed and it should prove to be a considerable asset, both as a textbook and as a book of reference to all those for whom it is intended.

J. H. P.

An experimental introduction to the theory of probability. By J. E. Kerrich. Pp. 98. Danish Cr. 8·50. 1946. (Munksgaard, Copenhagen).

The author, Senior Lecturer in the department of Mathematics of the University of the Witwatersrand, was caught in Denmark at the outbreak of war. He was interned by arrangement by the Danes at Hald, and " has great pleasure in thanking the Danish authorities for the measure of protection they were able to afford him, and in congratulating them on the truly admirable manner in which they cared for their internees for so many years ". Every year he lectures on elementary statistics : it is not clear whether this applies to the internment camp.

The treatment of his book starts with grounds common alike to mathematician and non-mathematician and builds up work to serve as an introduction to textbooks on probability. He does this by considering the results of two experiments.

(1) He spun, under stated conditions, a coin (apparently a Danish crown) ten thousand times, and noted the results. There were 5067 " heads ". A specimen record of two thousand spins is given. It is a pity that this table was not printed in more legible form, say in blocks of 5×5, separated by spaces or rules. A patch of this table satisfies fairly well the four criteria laid down by Kendall and Smith (*Journal of the Royal Statistical Society*, 1938, CI, i, p. 154) as far as they are appropriate to scale-of-two digits. The writer's own conclusion is (p. 25) that he is " by no means convinced that $p = \frac{1}{2}$ for our particular coin ".

A " second coin experiment " is referred to on p. 56. This appears to be that of pp. 52, 53, of spinning a coin twice, and the figures given seem to be those obtained by taking the ten thousand spins in twos, as we are given the results as *TT*, 1219 ; *HT*, 1194 ; *TH*, 1301 ; *HH*, 1286. These are, of course, what we use in applying to a head-tail case the Kendall-Smith serial test.

(2) Four ping-pong balls, two of one make and two of another, appearing " to be practically identical ", were shaken in a box and two drawn in succession. Five thousand drawings were made by Mr. Eric Christensen, a fellow-internee, and the following results were recorded : *RR*, 756 ; *RG*, 1689 ; *GR*, 1688 ; *GG*, 867 (where red and green were the distinguishing marks of the two brands). The results of five hundred consecutive drawings are given in detail. The author's conclusion about these (p. 98) is that " it does suggest rather forcibly that the red balls are not identical with the green ".

The tables are not set out in more detail. The results are tabulated on the whole to indicate the way that the various ratios seem to approximate to some value as the number of experiments increases.

On the data the author builds up his development of the theory, that is, the idea of probability, the addition theorem (" either . . . or " case, for mutually exclusive events), the multiplication theorem (" as well as " case, for compound events), of stochastic independence, of the binomial distribution, and of the normal curve. He makes the fundamental hypothesis of the truth of the impression that if we could take an unlimited succession of increasing values of n, and observe a large enough set of values of m/n for each value of n, then the successive sets would continue to cluster progressively closer together about some fixed value. He attempts to give more precision to some of the ideas in this hypothesis, and develops gently in an elementary fashion so that the student will have nothing to unlearn. The book can therefore be recommended to those who have to give their students the preliminary ideas, and the procedure will probably be found to be of definite pedagogic worth. Occasionally a word or phrase might be modified to make the meaning more clear or precise. But by his treatment the difficulties of " equally probable " are avoided.

There is no index and no bibliography. The book is printed in Denmark, but only rarely have misprints slipped past the eye of the proof-reader, some in the spelling of English words, and some in the use of mathematical symbolism.

FRANK SANDON.

On the principles of statistical inference. By A. WALD. Pp. ii, 47, ii. \$1. 1942. Lithoprinted. Notre Dame Mathematical Lectures, 1. (Notre Dame, Indiana)

This pamphlet contains four lectures given at the University of Notre Dame, Indiana, in 1941. It outlines the three special cases of statistical inference :

(1) The Neyman-Pearson theory of testing a statistical hypothesis involving, *e.g.*, the idea of the null hypothesis ;

(2) the R. A. Fisher theory of Statistical Estimation, including the ideas of maximum likelihood and efficiency ;

(3) the Neyman theory of Inference using the idea of Confidence Intervals.

Wald generalises the two problems—testing a hypothesis and estimation—and introduces ideas of " asymptotically most powerful tests " and of " statistical decision functions ", and puts forward certain suggestions for principles in connection with them.

FRANK SANDON.

Galois Theory. By E. ARTIN. 2nd edition. Pp. 82. \$1.25. 1944. Notre Dame Mathematical Lectures, 2. (Notre Dame, Indiana)

This is the second edition of Artin's booklet on the Galois theory. The first edition has been revised and augmented; in particular, an axiomatic account of determinants has been added to the first chapter. As the book has not yet been reviewed in the *Gazette*, the following remarks may not be out of place.

The author develops the subject axiomatically from the most modern point of view, and his treatment breathes the spirit of modern algebra. It is original and a great improvement on the classical theory. It presupposes, however, some maturity of thought on the part of the reader, and also some special knowledge, *e.g.* the elements of group theory. The book is a most interesting and useful one, more particularly to those who have some previous knowledge of the subject, since abstract reasoning is not easily appreciated by a beginner. The three chapters of which the book is composed contain a great variety of material and results and put in its proper light a subject which is of fundamental importance in many parts of algebra, both old and modern, as well as in number theory. The book can be strongly recommended to all those interested in Galois theory.

The first chapter deals with vector spaces and the linear dependence of vectors and this plays a very important part in the development of his account.

The second chapter deals with polynomials in fields and the extension fields in which the polynomial is reducible. The fundamental theorem of algebra, that every equation has a root, really a theorem in complex function theory, is replaced by the algebraic theorem of Kronecker, that if $f(x)$ is a polynomial in a field F, there exists an extension of F in which $f(x)$ has a root. It easily follows that there exists a splitting field of $f(x)$, that is, one in which $f(x)$ splits into linear factors. Then the Galois theory is developed in the simplest and most natural way by considering the isomorphisms and automorphisms of fields.

The third and last chapter is by A. N. Milgram and contains applications of the preceding theory to the solution of equations by radicals, *e.g.* a proof that the polynomial equation $f(x)=0$ is solvable by radicals if and only if its Galois group is solvable, and that the general equation of degree n is not solvable by radicals if $n>4$.

L. J. MORDELL.

The Teaching of Elementary Mathematics. By the late C. GODFREY and A. W. SIDDONS. Pp. xi, 322. 7s. 6d. 1946. (Cambridge University Press)

The second edition of this book comes most opportunely at a time when

" men and women from all walks of life and of widely differing ages are entering the recently opened Emergency Training Colleges ". Mr. Siddons is to be congratulated on his decision to make no changes in the original text which is, in fact, an ordered statement of the aims, ideals and methods of two great teachers of mathematics who played a prominent part in the reform of mathematical teaching. To have made minor changes would have added little to the inspirational value of the book; radical changes would perhaps have robbed us of the personalities of Godfrey and Siddons.

In reading this book again after a number of years one realises that although many of the reforms suggested in it have already been widely adopted some of its suggestions still await adoption. The first section dealing with that ever-present topic, " The place of Mathematics in Education " is an essay written by the late Professor Godfrey in 1911. Some passages of this essay read as though they were contributions to a recent discussion of the Mathematical Association. The " Tit-Bits " point of view is attacked and we are urged to relate our teaching to the knowledge of the child. The condemnation of excessive manipulative work in Algebra has an up-to-date ring. The essay is fresh, lively and stimulating throughout.

The second part of the book, written by Mr. Siddons, deals with " General Teaching Points " and with detailed methods of teaching the various branches of school mathematics. In the more general Part II the student may find guidance on planning a lesson, mistakes and corrections, the use and abuse of scrap paper, blackboard work and other equally important matters. In the later parts the teacher has a reference book of detailed methods of teaching fractions, decimals, areas and volumes and many other topics. As Mr. Durell says in his review of the first edition (*Mathematical Gazette*, Vol. XVI, No. 218), " These chapters . . . are probably not intended to be read at a sitting; they form a work of reference to which teachers may turn with advantage when wishing to find an account of a special theme ". This section might well have been dogmatic, but it is not. Mr. Siddons discusses methods employed by other teachers and gives reasons for his preferences.

Every student in training and teachers not familiar with the book should obtain a copy and study it carefully. The last word on mathematical teaching will never be said. " New methods must be tried if stagnation is to be avoided." One feels that these two pioneers Godfrey and Siddons would not wish for anything better than that teachers of the future should catch that spirit of adventure which was their own inspiration. F. J. S.

A Text Book of Elementary Astronomy. By E. A. Beet. Pp. x, 110, with 92 figs. 8s. 6d. 1945. (Cambridge University Press)

Mr. Beet's little book is based upon lessons in elementary astronomy conforming to the longer of two syllabuses suggested by the Science Masters' Association in 1938. It approaches the subject from the experimental standpoint, combining instruction in Astronomy with a first course in Light. This method should increase the popularity of both subjects in the schools and may well remove some of the difficulties of imparting the elements of astronomy to the younger generation—difficulties hitherto evaded by leaving them in total ignorance of the universe they live in. More than three-quarters of the book is devoted to the motions of the Earth and planets and a general account of the solar system; into the remainder is condensed a short account, from the physical side, of the Sun, stars and stellar universe. This arrangement is perhaps no more unbalanced than that of other recent books which have tended in the opposite direction, and can be defended with rather more justification in a book intended for schools.

A word of praise must be reserved for the refreshing series of more than 180

questions which conclude the book, and for the good selection of plates, excellently reproduced. A. H.

Analytic Geometry. By FREDERICK S. NOWLAN. 3rd edition. Pp. 355. \$2.75. 1946. (McGraw Hill)

This is a new and revised edition of an elementary text on two and three dimensional solid geometry. It is designed for classes of which many members are not expected to specialise in Mathematics, and some may have no further mathematical instruction. The author has attempted, with considerable success, to provide these students with some effective mathematical training, and has developed the subject with great care and attention to detail, his treatment being accordingly much more rigorous than the usual presentations.

The book has, besides, a number of original aspects of which two are worthy of special notice. The study of plane geometry is based on the use of direction cosines. This method is not without its advantages, and seems to facilitate the development of the solid geometry. The author's treatment of polar coordinates is also original, and will perhaps assist some students in their understanding of them.

Teachers in this country will find that, except for the final chapters on solid geometry, the presentation is more advanced than the subject-matter, (there is, for instance, no mention of oblique axes), and, as is usual in American textbooks, that the examples are rather too easy. D. B. S.

How to solve it. By G. PÓLYA. Pp. 204. 16s. 6d. 1945. (Princeton University Press ; Humphrey Milford)

This is a book for students and teachers : primarily for students and teachers of mathematics, but it might interest any educated teacher of any subject which does not consist in the mere accumulation of knowledge. Its appeal would perhaps be greatest at the Training College level, and although it contains very little that a good teacher would not discover for himself, it might very well accelerate the process of discovery. The central idea is a set of questions and suggestions very clearly set out under the headings " understanding the problem ", " devising a plan ", " carrying out the plan ", and " looking back ". It is arranged in three parts : I. In the classroom, II. How to solve it (a dialogue between teacher and student), III. A dictionary of heuristic ; and it is part III that occupies the bulk (80%) of the volume. There are 64 entries in the dictionary which is, as the author says, often condensed and sometimes subtle, so that it needs careful reading. Under the entry Bolzano, that writer is quoted as saying, " I shall take pains to state in clear words the rules and ways of investigation which are followed by all able men, who in most cases are not even conscious of following them . . . I hope that the little that is presented here may please some people and have some application afterwards ". If, as may be suspected, Prof. Pólya has similar hopes to Bolzano, he is likely not to be disappointed.

Among the more substantial articles in the dictionary are : Analogy, Definition, Induction, Heuristic, Notation, Pappus, and Reductio ad Absurdum. In this last article Euclid's famous theorem IX, 20 is given, though not quite in Euclid's form. It is not always realised how beautifully Euclid expressed the matter, although Hardy's *Apology* selects this theorem as one of two simple and first-rate real mathematical theorems. Prime numbers, Euclid said, are more than any assigned multitude of prime numbers. The essence of the proof is that if x is the product of the assigned numbers, $x+1$ is either a prime or is divisible by a prime that is not one of the assigned ones. Stated like that, it does not seem to be an illustration of *reductio ad absurdum*

so much as a direct recipe for producing another prime number. But it is true that Euclid does appeal to the reductio principle at one stage of his proof.

Among the slighter entries in the dictionary may be mentioned the two rules of discovery: have brains and good luck, and sit tight till you get a bright idea; and also the two rules of style: have something to say, and do not say it twice. A. R.

The Gyroscope and its Applications. Edited by MARTIN DAVIDSON. In three sections. Section I: General Theory, by M. DAVIDSON. Section II: Marine Applications, by G. C. SAUL. Section III: Aeronautical Applications, by J. A. WELLS and A. P. GLENNY. Pp. 256. 21s. 1946. (Hutchinson's Scientific and Technical Publications)

The three sections into which this book is divided occupy 37, 72 and 112 pages respectively. There are also appendices to Section I (18 pages) and a glossary of 4 pages. The readers of the *Mathematical Gazette* will naturally be most interested in the first section, in which the author attempts to explain the theory of the gyroscope in simple language.

The success or failure of a new attempt to explain the gyroscope depends on what effect the attempt has on a reader who does not know how the gyroscope works, and it is not easy for a reviewer who has a certain familiarity with the ideas to put himself in the place of the beginner. However, the effort must be made, and it is my opinion that the author has been successful in his main object and that the thirty-seven pages of Section I will give to those to whom the gyroscope is just a mystery a very good working idea of its method of operation. Incidentally they will also have learnt the principle of the operation of the gyro-compass, that "mystery within a mystery", whose explanation frequently baffles many a mathematician who is familiar with the ordinary equations of the top. The explanation of precession, the central point of difficulty in explaining the gyroscope, is admirably lucid and illustrated by lettered photographs of an actual gyroscope. Numerical illustrative examples are incorporated in the text, and the mathematical equipment required of the reader is very modest.

So much on the credit side: I must now refer to several blemishes, which might, I think, have been avoided with a little more care. (1) It was surely unnecessary to spend half a page explaining the radian, two pages on moments of inertia, and half a page on the parallelogram of velocities. I cannot conceive that anyone who could profitably study this book would be ignorant of these notions. (2) The author cannot, apparently, make up his mind whether he wishes to use absolute or gravitational units, nor what he wishes to call them. For instance, angular momentum is given in terms of ft.pdls.-sec. and ft.pds.-sec. in consecutive sentences on p. 39 (though it is true that on p. 34 ft. pd.sec. has been replaced in the errata slip by lb.ft.^{2}/sec.). (3) Angular momentum or moment of momentum? It is probably immaterial which term is used, but it would be less confusing if the author made up his mind which term he preferred and then stuck to his opinion. (4) It is possible to take up the position that the only thing that matters in a scientific work is the content and that the manner of presentation and, in particular, the grammar and style of the language used, are beyond criticism. Those who do not take this view will regret that the author did not obtain the services of a competent English scholar to read and criticise the manuscript before publication. As an illustration, I quote a sentence (p. 20): "Suppose we calculate the angular momentum of a gyroscope about a line drawn from the instrument to a star, given the mass of the wheel, its dimensions, and rate of spin, and, to simplify the problem, we can imagine that this line coincides with the axle of the wheel". (5) The explanation of the balancing of a bicycle (p. 41) is confused

and misleading, though here the author errs in company, for it is not easy to find a sound explanation in any book. (6) On p. 15, Fig. I, 1(a) seems to contradict the letterpress. There appears to be no reason why the toe of the top should move on the surface at all. Another point here is that the arrows purporting to indicate the direction of rotation of the tops are entirely ambiguous; the tops might be spinning either way. (7) On p. 34 the remarkable result is established that the radius of gyration of a rod about an axis through its centre is 2·3 times its length. (8) In the problem on p. 39, what has the "set screw" to do with things? Surely some explanation is required.

In addition to the above, the following minor misprints, etc., in Section I have been noted:

P. 28, l. 2—"outer" should be "middle" (this is a subtle mistake. I read the next two pages three times before discovering what was wrong!).

P. 33, 12 lines from end—clear to the expert, but possibly not to the beginner, that the denominator of each fraction is 3 only, *i.e.* $1/3\mu l^2$ means "one-third of μl^2".

P. 37, 3 lines from end—"density" should be "specific gravity".

P. 40, l. 17—what is it that weighs 35 lb.? The English would appear to mean the motor cycle, but later the machine and rider are said to weigh 300 lb., which would give a rider of 19 stone—rather hard on the ultra-light-weight motor-bike!

While Section I is written for the beginner, Sections II and III are written for the practical man—sailor or airman—who knows a good deal about machinery and ships or aeroplanes, and wishes to know about the applications of the gyroscope. At any rate, this is the conclusion arrived at on reading these sections and noticing the continual use of technical terms, which are, no doubt, familiar to this class of readers. Whether these sections really fulfil their purpose, I am not expert enough to judge. The editor himself states that many applications have had to be omitted for security reasons. Even so, he and his collaborators have certainly covered a wide field. Section II deals mainly with gyro-compasses and gyro-stabilisers, and Section III with direction indicators, rate-of-turn indicators, gyro-verticals, and auto-pilots. As an amateur critic, I would say that these sections seem very adequate, but rather detailed in parts.

To the academical mathematician or scientist, including the average school-teacher or sixth-form mathematical or physical specialist, the detailed study of Sections II and III, with a proper understanding of all the words mentioned, would be a difficult task. Nevertheless, by skimming through these sections, he can obtain an insight into the practical applications of the gyroscope which is very fascinating.

Finally, the Appendices to Section I. These comprise several pieces of mathematics which refer to subjects treated in Sections II and III, such as "Speed and Course Error", "Period of undamped Oscillation of the Gyro-compass Axle in Azimuth", etc. It is, to say the least, confusing to have this work placed in the position it is, but, apart from this, it seems quite foreign to the spirit of the book for it to be there at all. In my opinion the book would be better without these Appendices.

If this book were drastically overhauled it might become a very good book indeed. As it is, at 21s. it would be a purchase well worth while for a school library. Section I would be a real help to the young mathematician or physicist or engineer in understanding the gyroscope and Sections II and III a valuable stimulus to future activities. F. G. MAUNSELL.

Scientific Instruments. Edited by H. J. COOPER. Pp. vii, 293. 25s. 1946. (Hutchinson)

The underlying idea of this attractive book is novel, being an attempt to

satisfy the curiosity of scientific workers about the instruments used in other branches of science than their own. A great variety of instruments are described, from spectroscopes to gas meters, and from thermionic valves to clocks. The twenty-nine chapters are covered in all by fifteen authors including the editor, the book being divided for convenience into five main sections.

The first section is optical and deals fairly comprehensively with lenses, cameras, microscopes, polarimeters, photometers, rangefinders, refractometers, interferometers, spectroscopes and telescopes. The descriptions on the whole are clear, but one is conscious of a fundamental difficulty in writing a book of this type. It is not easy to give a convincing explanation of the working of an instrument without at the same time going into the theory upon which it is based.

The next section is described by the comprehensive title of measuring instruments. The chapter headings include such diverse subjects as density, electrical instruments and speed. Many of these instruments are familiar in lower school laboratories.

Navigational and surveying instruments are covered in the third section, which includes a chapter on gyroscopic instruments in general. The author of this chapter has at one or two points failed to combine accuracy with simplicity. For instance we are told that *any* body set spinning will maintain the direction of its axis of rotation as long as it is undisturbed! A rather more careful choice of wording would be sufficient to convey the required meaning, and would not at the same time conflict with Euler's equations.

A short section on liquid testing for viscosity precedes the last section, which contains chapters on miscellaneous topics. The first of these chapters deals with acoustics, and the second covers calculating machines, including slide rules. This is well written as far as it goes. Finally a chapter on hardness testing is followed by one on vacuum tubes. The latter gives a clear account of the internal working of thermionic valves, but is very cursory on their applications.

The above criticisms are in the nature of blemishes rather than serious faults. The book does in fact contain much useful and interesting material, and can be recommended for a place on the shelves of any scientific or technical library.

B. M. B.

BOOKS RECEIVED FOR REVIEW.

G. W. Scott Blair. *A survey of general and applied rheology.* Pp. xvi, 196. 18s. 6d. 1946. (Pitman)

C. Chevalley. *Theory of Lie groups. I.* Pp. xii, 217. 20s. 1946. (Princeton University Press; Oxford University Press)

F. M. Colebrook. *Basic mathematics for radio students.* Pp. x, 270. 10s. 6d. 1946. (Iliffe)

H. G. Lieber and L. R. Lieber. *Modern mathematics for T.C.Mits.* Pp. 230. 7s. 6d. 1946. (Allen & Unwin)

H. O. Newboult. *Analytical method in dynamics.* Pp. 81. 7s. 6d. 1946. (Oxford University Press)

British Association Mathematical Tables.

Part-volume A: *Legendre polynomials.* Prepared by L. J. Comrie. Pp. 42. 8s. 6d. 1946.

Part-volume B: *The Airy integral, giving tables of solutions of the differential equation $y''=xy$.* Prepared by J. C. P. Miller. Pp. 56. 10s. 1946.

Auxiliary Tables, I: *Coefficients in the modified Everett interpolation formula.* 6d. 1946.

Auxiliary Tables, II: *Table for interpolation with reduced derivatives. Coefficients for function and for first derivative.* 6d. 1946. (Cambridge University Press)

Select list of standard British scientific and technical books. 3rd edition. Pp. 63. 5s. 1946. (Aslib, 52 Bloomsbury Street)

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Adiecta præfatiuncula in qua de disciplinis Mathematicis nonnihil.

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